ISRAEL JOURNAL OF MATHEMATICS, Vol. 63, No. 1, 1988

THE INVARIANT SUBSPACE PROBLEM FOR A CLASS OF BANACH SPACES, 2: HYPERCYCLIC OPERATORS

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ABSTRACT

We continue here the line of investigation begun in [7], where we showed that on every Banach space $X = l_1 \oplus W$ (where W is separable) there is an operator T with no nontrivial invariant subspaces. Here, we work on the same class of Banach spaces, and produce operators which not only have no invariant subspaces, but are also hypercyclic. This means that for every nonzero vector x in X, the translates $T^r x$ $(r = 1, 2, 3, \ldots)$ are dense in X. This is an interesting result even if stated in a form which disregards the linearity of T: it tells us that there is a continuous map of $X \setminus \{0\}$ into itself such that the orbit $\{T^r x : r \geqq 0\}$ of any $x \in X \setminus \{0\}$ is dense in $X \setminus \{0\}$. The methods used to construct the new operator T are similar to those in [7], but we need to have somewhat greater complexity in order to obtain a hypercyclic operator.

§1. Introduction

Throughout this paper the underlying field for our Banach spaces may be either **R** or **C**.

As we did in [7], we may without loss of generality assume that W is equipped with a biorthogonal system $\{x_i\}_{i=1}^{\infty}$, $\{x_i^*\}_{i=1}^{\infty}$ such that $\overline{\text{lin}}\{x_i\} = W$ and $\| x_i \| = \| x_i^* \| = 1$ for each i. We shall write $(g_i)_{i=0}^{\infty}$ for the unit vector basis of l_1 and we shall assume that the direct sum $l_1 \oplus W$ is taken in the sense of l_1. Writing F for the dense subspace of X spanned by $\{x_i\} \cup \{g_i\}$ the norm of an operator $S : F \to Z$ is bounded by

$$(1.1) \qquad \| S \| \leqq \left(\sup_{i \geqq 0} \| Sg_i \| \right) \vee \left(\sum_{i=1}^{\infty} \| Sx_i \| \right).$$

Received July 16, 1987 and in revised form March 19, 1988

1.2. Let $\boldsymbol{d} = (d_i)_{i=1}^{\infty}$ denote a strictly increasing set of positive integers, which will be required to "increase sufficiently rapidly" in the usual sense (see [4], §1). We will split up $\boldsymbol{d}$ into three subsequences $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, where $a_i = d_{3i-2}$, $b_i = d_{3i-1}$ and $c_i = d_{3i}$. So $a_1 < b_1 < c_1 < a_2 < b_2 < \cdots$. Let us also write $h_i = \lceil 2 \log_2 c_i \rceil$ (where $\lceil x \rceil$ denotes the least integer not less than x).

1.3. Once again, we use the abbreviation p.$\boldsymbol{d}$. to mean "provided $\boldsymbol{d}$ increases sufficiently rapidly".

1.4. We write $|p|$ for the sum of the absolute values of the coefficients of the polynomial p, deg p for the degree of p, and $|p|_0$ for the largest absolute value of any coefficient.

1.5. We define the sequence $(f_i)_{i=0}^{\infty}$ as in [1], §1.6;

$$f_i = \begin{cases} x_n, & \text{if } i = a_n - 1, \quad n \in \mathbf{N}; \\ g_i, & \text{if } 0 \leqq i < a_1 - 1; \\ g_{i-n}, & \text{if } a_n \leqq i < a_{n+1} - 1, \quad n \in \mathbf{N}. \end{cases}$$

1.6. If $S \subset \{g_i : i \geqq 0\}$ we write π_S for the norm 1 projection $X \to \overline{\mathrm{lin}}(S)$ which annihilates W and sends g_i to g_i $(i \in S)$ or zero $(i \notin S)$.

§2. Defining the operator

2.1. Definition. For each integer $\nu \geqq 0$, let $(P_\nu, |\cdot|)$ be the finite dimensional Banach space of polynomials of degree at most ν, with the $|\cdot|$ norm given by (1.4), and let us choose once and for all a minimal ε-net π_ν for the unit ball of P_ν, where $\varepsilon = 4^{-\nu}$, and let us write $M(\nu)$ for the number of elements in π_ν. We will write $\pi_\nu = \{\bar{p}_{i,\nu} : 1 \leqq i \leqq M(\nu)\}$.

2.2. Definition. Given the sequence $\boldsymbol{d}$, let us define three further sequences $(\mu_i)_{i=1}^{\infty}$, $(\nu_i)_{i=1}^{\infty}$ and $(\xi_i)_{i=0}^{\infty}$, as follows.

$$\xi_0 = 0, \tag{2.2.1}$$

$$\mu_n = n a_n, \tag{2.2.2}$$

$$\nu_n = n(a_n + b_n), \tag{2.2.3}$$

$$\xi_n = \nu_n + h_n \sum_{j=1}^{M(\nu_n)} c_n^j. \tag{2.2.4}$$

2.3. Definition. Given $\boldsymbol{d}$, let Λ_n be the collection of all nonzero polynomials $p(t)$ with the following properties.

(2.3.1) p has nonnegative integer coefficients.

(2.3.2) $$|p|_0 \leqq h_n,$$

(2.3.3) $$\deg p \leqq M(\nu_n),$$

(2.3.4) $$t \mid p(t).$$

2.4. Definition. For each n we order $\Lambda_n \cup \{0\}$ lexicographically by writing $p_1 < p_2$ if, when we write

$$p_i(t) = \sum_{j=1}^{M(\nu_n)} \alpha_{ij} t^j \qquad (i = 1, 2),$$

we have $\alpha_{1J} < \alpha_{2J}$ where $J = \max\{j : \alpha_{1j} \neq \alpha_{2j}\}$.

The reason why this is the "correct" ordering for Λ_n is given by the following simple lemma.

2.5. Lemma. P.$\boldsymbol{d}$. *the following is true.*
For all $n \in \mathbf{N}$, *and all* $p_1, p_2 \in \Lambda_n \cup \{0\}$ *such that* $p_1 < p_2$, *we have*

(2.5.1) $$p_1(c_n) \leqq p_2(c_n) - c_n;$$

also,

(2.5.2) $$p_1(c_n) < p_2(c_n) - \tfrac{3}{4} c_n^J$$

where $J = \max\{j : \alpha_{2j} \neq \alpha_{1j}\}$ (notation as in 2.4).

Proof. With the notation of §2.4,

$$\begin{aligned} p_2(c_n) - p_1(c_n) &= \sum_{j=1}^{M(\nu_n)} (\alpha_{2j} - \alpha_{1j}) c_n^j \\ &= \sum_{j=1}^{J} (\alpha_{2j} - \alpha_{1j}) c_n^j \qquad \text{(by definition of } J\text{)} \\ &\geqq c_n^J - |p_1| \cdot c_n^{J-1} \\ &\geqq c_n^J - \deg p_1 \cdot |p_1|_0 \cdot c_n^{J-1} \end{aligned}$$

$$\geqq c_n^J - M_1(v_n)\cdot\lceil 2\log_2 c_n\rceil\cdot c_n^{J-1} \qquad \text{(by 2.3)}$$
$$\geqq \tfrac{3}{4}\cdot c_n^J \qquad \text{p.}\boldsymbol{d}.,$$

since $M(v_n)$ depends only on n, a_n and b_n. So (2.5.2) holds; (2.5.1) follows trivially because $p_1(t)$ and $p_2(t)$ are both divisible by t.

2.6. DEFINITION. We write $\hat{p}_n$ for $\max(\Lambda_n, <)$, and we note that

$$\hat{p}_n(t) = h_n \sum_{k=1}^{M(v_n)} t^k. \tag{2.6.1}$$

If $p < \hat{p}_n$, $p \in \Lambda_n \cup \{0\}$, we write p^+ for the successor of p in $(\Lambda_n \cup \{0\}, <)$, and $J(p)$ for the largest k such that the coefficients of t^k in $p(t)$ and $p^+(t)$ are different.

2.7. DEFINITION. Let E be a vector space of countable dimension over our underlying field, with basis $(e_i)_{i=0}^{\infty}$. Let $T: E \to E$ be the right shift, that is, the linear map such that $T(e_i) = e_{i+1}$ for each i.

We now come to the main definition of the proof.

2.8. DEFINITION. Let the sequence $\boldsymbol{d}$ be given. We shall show that, p.$\boldsymbol{d}$., there is a unique isomorphism $\theta: F \to E$ with the following properties.

(2.8.0) $\theta f_0 = e_0$.

(2.8.1) If integers r, n, i satisfy $0 < r \leqq n$, $i \in [0, \xi_{n-r}] + ra_n$ then

$$\theta f_i = a_{n-r}(e_i - e_{i-ra_n}) \qquad \text{(we take } a_0 = 1\text{).}$$

(2.8.2) If integers r, n, i satisfy $1 < r \leqq n$,
$i \in ((r-1)a_n + \xi_{n-r+1}, ra_n)$ (respectively, $n \geqq 1$, $i \in (\xi_{n-1}, a_n)$), then

$$\theta f_i = 2^{(h-i)/\sqrt{a_n}}\cdot e_i,$$

where $h = (r - \tfrac{1}{2})a_n$ (respectively, $h = \tfrac{1}{2}a_n$).

(2.8.3) If integers r, n, i satisfy $1 \leqq r \leqq n$, $i \in [r(a_n + b_n), na_n + rb_n]$, then

$$\theta f_i = e_i - b_n e_{i-b_n}.$$

(2.8.4) If integers r, n, i satisfy $1 \leqq r \leqq n$, $i \in (na_n + (r-1)b_n, r(a_n + b_n))$ then

$$\theta f_i = 2^{(h-i)/\sqrt{b_n}}\cdot e_i,$$

where $h = (r - \tfrac{1}{2})b_n$.

(2.8.5) If for some $p \in \Lambda_n$ we have $i \in [p(c_n), p(c_n) + \nu_n]$, let $d = \deg p$. Then

$$\theta f_i = 2^{1-|p|} \cdot c_n (e_i - \bar{p}_{d,\nu_n}(T) e_{i-c_n^d}).$$

(2.8.6) If $p < \hat{p}_n$, $p \in \Lambda_n \cup \{0\}$ and $i \in (p(c_n) + \nu_n, p^+(c_n))$ then

$$f_i = 2^{(h-i)/c_n^{J(p)-1/2}} \cdot e_i,$$

where $h = (p(c_n) + p^+(c_n))/2$.

Note that, p.$\boldsymbol{d}$., Definition 2.8 gives $\theta f_i = \Sigma_{j=0}^{i} \alpha_{ij} e_j$ uniquely for each $i \geqq 0$, and since α_{ii} is never zero, this linear relationship is invertible. So θ exists and is an isomorphism, and indeed,

$$\operatorname{lin}\{e_i : 0 \leqq i \leqq n\} = \theta \operatorname{lin}\{f_i : 0 \leqq i \leqq n\} = F_n,$$

say, for all $n \geqq 0$. We shall identify E with the dense subspace F of X by $\theta^{-1} : E \to F$. If $x = \Sigma_{i=0}^{N} \lambda_i e_i \in F$, we write $|x| = \Sigma_{i=0}^{N} |\lambda_i|$.

2.9. Definition. For each $n \geqq 0$, let $I_n : (F_n, \|\cdot\|) \to (F_n, |\cdot|)$ be the identity map. We note that if $n = \mu_m$ (respectively, $n = \nu_m$ of $n = \xi_m$) for some $m \in \mathbf{N}$, then the norm $\|\cdot\|$ on F_n depends only on the underlying space X, the value of m, and the elements $\{a_i : 1 \leqq i \leqq m\}$, $\{b_i, c_i : 1 \leqq i \leqq m-1\}$ (respectively, $\{a_i, b_i : 1 \leqq i \leqq m\}$, $\{c_i : 1 \leqq i \leqq m-1\}$ or $\{a_i, b_i, c_i : 1 \leqq i \leqq m\}$) of the sequence $\boldsymbol{d}$. So for a given space X, there are functions $M_1, M_2, M_3 : \mathbf{N}^2 \to \mathbf{N}$ such that for all $m \in \mathbf{N}$, and all $\boldsymbol{d}$ such that (2.8) is meaningful, we have

(2.9.1) $$\|I_{\mu_m}\| \vee \|I_{\mu_m}^{-1}\| \leqq M_1(m, a_m),$$

(2.9.2) $$\|I_{\nu_m}\| \vee \|I_{\nu_m}^{-1}\| \leqq M_2(m, b_m),$$

(2.9.3) $$\|I_{\xi_m}\| \vee \|I_{\xi_m}^{-1}\| \leqq M_3(m, c_m).$$

Let us choose such functions, once and for all.

2.10. Definition. Let Q_m $(m \geqq 1)$ denote the projection $F \to F_{\mu_m}$ such that

$$Q_m(f_j) = \begin{cases} f_j, & 0 \leqq j \leqq \mu_m, \\ f_{j - ra_n + (r-n+m)a_m}, & j \in [0, \xi_{n-r}] + ra_n; \quad 0 < n - m < r \leqq n, \\ 0, & \text{otherwise.} \end{cases}$$

2.11. Definition. Let R_m^0 $(m \geqq 1)$ denote the projection $F \to F_{\xi_m}$ such that

$$R_m^0(f_j) = \begin{cases} f_j, & 0 \leqq j \leqq \xi_m, \\ -a_{n-r}e_{j-ra_n}, & j \in [0, \xi_{n-r}] + ra_n, \quad 0 < n-m < r \leqq n, \\ 0, & \text{otherwise,} \end{cases}$$

and let $Q_m^0 = (I - \pi_s) \circ R_m^0$ where $S = \{f_i : \mu_m < i \leqq \xi_m\} \subset \{g_i\}$.

2.12. Definition. Let $P_{n,m}$ $(m > n \geqq 1)$ be the operator $\tau_{n,m} \circ Q_m$, where $\tau_{n,m} : F_{\mu_m} \to F_{\mu_m}$,

$$\tau_{n,m}(e_j) = \begin{cases} e_j, & 0 \leqq j < (m-n)a_m, \\ 0, & (m-n)a_m \leqq j \leqq ma_m. \end{cases}$$

Note 2.13. The map $T : e_i \to e_{i+1}$ may now be regarded as a map on F. Our job is to show that, p.$\boldsymbol{d}$., this operator is continuous and extends continuously to a hypercyclic operator on X.

§3. Preliminary arguments

Lemma 3.1. *$\| Q_m \| \leqq m$ for each m.*

Proof. By (1.1), $\| Q_m \| \leqq \max(\sup_i \| Q_m g_i \|, \Sigma_{i=1}^{\infty} \| Q_m x_i \|)$. Now (2.10) gives $Q_m(f_j) = f_i$ (some $i \leqq j$) or zero for every j; so $\| Q_m f_j \| \leqq 1$ hence $\| Q_m g_j \| \leqq 1$ for all j. Thus, $\| Q_m \| \leqq \max(1, \Sigma_{i=1}^{\infty} \| Q_m x_i \|)$. But $Q_m x_i = Q_m f_{a_i - 1} = f_{a_i - 1}$ $(i \leqq m)$ or zero $(i > m)$ (3.1.1). Hence, $\| Q_m \| \leqq m$.

Lemma 3.2. *$\| Q_m^0 \|$, $\| R_m^0 \| \leqq a_m$ for all m,* p.$\boldsymbol{d}$.

Proof. Definition 2.11 gives $R_m^0(x_i) = x_i$ $(i \leqq m)$ or 0 $(i > m)$, so

$$\| R_m^0 \| \leqq \max(\sup \| R_m^0 g_i \|, m).$$

Now $\{g_i\} \subset \{f_i\}$, and $R_m^0(f_j)$ is f_j or zero unless $j \in [0, \xi_{n-r}] + ra_n$, $0 < n - m < r \leqq n$, when

$$\| R_m^0 f_j \| = a_{n-r} \| e_{j-ra_n} \| \leqq a_{m-1} \| e_{j-ra_n} \| \leqq a_{m-1} \| I_{\xi_{m-1}}^{-1} \| \, | e_{j-ra_n} |$$

(since $j - ra_n \leqq \xi_{m-1}$)

$$\leqq a_{m-1} M_3(m-1, c_{m-1}) \qquad \text{by (2.9.3).}$$

Hence

$$\| R_m^0 \| \leqq \max(a_{m-1} M_3(m-1, c_{m-1}), m) \leqq a_m \qquad \text{p.}\boldsymbol{d}.$$

and

$$\| Q_m^0 \| \leqq \| I - \pi_S \| \, \| R_m^0 \| = \| R_m^0 \| .$$

LEMMA 3.3. *P.d., we have* $\| P_{n,m} \|, \| \tau_{n,m} \| \leqq \max(a_{n+1}, m)$, *and* $\| Q_n P_{n,m} \| \leqq a_{n+1}$ *for all* $n < m$.

PROOF. Using (1.1) again, we examine $\sup \| P_{n,m} g_i \|$ and $\Sigma \| P_{n,m} x_i \|$. Now

$$P_{n,m}(x_i) = \tau_{n,m} Q_m(x_i)$$

$$(3.3.1) \qquad = \begin{cases} \tau_{n,m} x_i, & i \leqq m \\ 0, & i > m \end{cases}$$

$$= \begin{cases} x_i, & i \leqq m, \\ 0, & i > m. \end{cases}$$

Hence

$$(3.3.2) \qquad \sum_i \| P_{n,m} x_i \| = m.$$

Now $\{g_i\} \subset \{f_i\}$, and $P_{n,m} f_i = \tau_{n,m} Q_m f_i$; and by (2.10), $Q_m f_i$ is either zero or f_j for some $j \in [0, ma_m]$. Thus

$$\sup_i \| P_{n,m} g_i \| \leqq \max_{0 \leqq j \leqq ma_m} \| \tau_{n,m} f_j \| .$$

Now (2.8) and (2.12) give for all $j \in [0, ma_m]$,

$$\tau_{n,m}(f_j) = \begin{cases} -a_{m-r} e_{j - ra_m}, & j \in [0, \xi_{m-r}] + ra_m, \quad m - n \leqq r \leqq m, \\ f_j \text{ or zero}, & \text{otherwise.} \end{cases}$$

Hence,

$$\max_j \| \tau_{n,m} f_j \| \leqq \max\left(1, \sup_{\substack{j \in [0, \xi_n] + ra_m \\ m - n \leqq r \leqq m}} a_{m-r} \| e_{j - ra_m} \| \right)$$

$$\leqq 1 \vee \left(a_n \max_{i \in [0, \xi_n]} \| e_i \| \right)$$

$$\leqq a_n M_3(n, c_n) \qquad \text{by (2.9.3).}$$

Hence

(3.3.3) $$\sup \| P_{n,m} g_i \| \leqq a_n M_3(n, c_n),$$

and given (3.3.2) we have

$$\| P_{n,m} \| \leqq \max(m, a_n M_3(n, c_n)) \leqq \max(m, a_{n+1}) \qquad \text{for all } n \quad \text{p.}\boldsymbol{d}.$$

Since Q_m acts as the identity on the domain of $\tau_{n,m}$, we have

$$\| \tau_{n,m} \| \leqq \| \tau_{n,m} \circ Q_m \| = \| P_{n,m} \| .$$

We now use (1.1) to examine the operator $Q_n P_{n,m}$. Let us consider first the sum $\Sigma_i \| Q_n P_{n,m} x_i \|$. By (3.3.1),

$$Q_n P_{n,m} x_i = \begin{cases} Q_n x_i, & i \leqq m \\ 0, & i > m \end{cases}$$

$$= \begin{cases} x_i, & i \leqq n, \\ 0, & i > n. \end{cases}$$

Thus $\Sigma_i \| Q_n P_{n,m} x_i \| = n$. By (3.3.3), we have

$$\sup_i \| Q_n P_{n,m} g_i \| \leqq \| Q_n \| \cdot a_n \cdot M_3(n, c_n) \leqq n a_n M_3(n, c_n)$$

by Lemma 3.1. Hence, by (1.1),

$$\| Q_n P_{n,m} \| \leqq \max(n, n a_n M_3(n, c_n))$$
$$\leqq a_{n+1} \quad \text{p.}\boldsymbol{d}.$$

This concludes the proof of Lemma 3.3.

§4. Continuity of T

LEMMA 4. *Let $\eta > 0$ be given. The following is true* p.$\boldsymbol{d}$. $\| T \| \leqq 1 + \eta$; *indeed,* $\| T |_{l_1 \cap F} \| \leqq 1 + \eta$ *while* $T |_{W \cap F}$ *has nuclear norm at most* η.

PROOF. By (1.1) it is sufficient to show that $\sup_i \| T g_i \| \leqq 1 + \eta$ while $\Sigma_i \| T x_i \| \leqq \eta$, since $\Sigma \| T x_i \|$ is obviously an upper bound not only for the norm of $T |_W$ but also for the nuclear norm. The second statement is easy to prove. By (1.5), (2.8),

$$x_i = f_{a_i - 1} = 2^{(1 - a_i/2)/\sqrt{a_i}} \cdot e_{a_i - 1};$$

$$Tx_i = 2^{(1-a_i/2)/\sqrt{a_i}} \cdot e_{a_i} = 2^{(1-a_i/2)/\sqrt{a_i}} \cdot (f_0 + a_{i-1}^{-1} f_{a_i}),$$

(4.0.1) $$\| Tx_i \| = (1 + a_{i-1}^{-1}) \cdot 2^{(1-a_i/2)/\sqrt{a_i}},$$

hence

(4.0.2) $$\sum_i \| Tx_i \| < \eta \quad \text{p.}\boldsymbol{d}.$$

The first statement, that $\sup_i \| Tg_i \| \leqq 1 + \eta$, is equivalent to showing that $\sup_i \| Tf_i \| \leqq 1 + \eta$, and this we prove by considering different values of i case by case. We use (2.8) freely to obtain the following.

Case 0. If $i = 0$ then $f_i = e_0$, $Tf_i = e_1 = 2^{(1-a_1/2)/\sqrt{a_1}} \cdot f_1$;

$$\| Tf_0 \| = 2^{(1-a_1/2)/\sqrt{a_1}} < 1 \quad p.\boldsymbol{d}.$$

Note that, to identify $f_1 = 2^{(a_1/2-1)/\sqrt{a_1}} \cdot e_1$ by (2.8.2) we assume that $1 \in (0, a_1)$, i.e. $a_1 > 1$. Such minor assumptions concerning the "rapid increase" of $\boldsymbol{d}$ will be made throughout this section: specifically, we shall assume that $\boldsymbol{d}$ increases so rapidly that all the intervals used in Definition 2.8 to split up the cases are nonempty.

Case 1. If $i \in [0, \xi_{n-r}] + ra_n$ $(0 < r \leqq n)$ and $i < \xi_{n-r} + ra_n$ then $Tf_i = f_{i+1}$. If $i = \xi_{n-r} + ra_n$ then

$$Tf_i = a_{n-r}(e_{1+\xi_{n-r}+ra_n} - e_{1+\xi_{n-r}}) = a_{n-r}(\varepsilon_1 f_{1+\xi_{n-r}+ra_n} - \varepsilon_2 f_{1+\xi_{n-r}}),$$

where

(4.1.1) $$\varepsilon_2 = \varepsilon_2(n, r) = 2^{(1+\xi_{n-r}-a_{n-r+1}/2)/\sqrt{a_{n-r+1}}}$$

and

(4.1.2) $$\varepsilon_1 = \varepsilon_1(n, r) = \begin{cases} 2^{(1+\xi_{n-r}-a_n/2)/\sqrt{a_n}}, & r < n, \\ 2^{(1+\mu_n-b_n/2)/\sqrt{b_n}}, & r = n. \end{cases}$$

P.$\boldsymbol{d}$., we have $\varepsilon_1 \vee \varepsilon_2 < (2a_{n-r})^{-1}$ for all $0 < r \leqq n$, hence $\| Tf_i \| \leqq 1$. Thus $\| Tf_i \| \leqq 1$ for all i covered by Case 1.

Case 2. If $i \in ((r-1)a_n + \xi_{n-r+1}, ra_n)$ with $1 < r \leqq n$ (respectively, $i \in (\xi_{n-1}, a_n)$, $n \in \mathbf{N}$), and if $i \neq ra_n - 1$ (respectively, $i \neq a_n - 1$), then $Tf_i = 2^{1/\sqrt{a_n}} \cdot f_{i+1}$, $\| Tf_i \| = 2^{1/\sqrt{a_n}}$. If $i = ra_n - 1$ $(1 \leqq r \leqq n)$ then

$$Tf_i = 2^{(1-a_n/2)/\sqrt{a_n}} \cdot e_{ra_n} = 2^{(1-a_n/2)/\sqrt{a_n}} \cdot (f_0 + a_{n-r}^{-1} f_{ra_n})$$

hence

$$\| Tf_i \| \leqq 2 \cdot 2^{(1-a_n/2)/\sqrt{a_n}} < 1 \quad \text{p.}\boldsymbol{d}.$$

Thus,

$$\| Tf_i \| \leqq 2^{1/\sqrt{a_n}} \leqq 2^{1/\sqrt{a_1}} \leqq 1 + \eta \quad \text{p.}\boldsymbol{d}.,$$

for all i covered by Case 2.

Case 3. (a) If $i \in [r(a_n + b_n), na_n + rb_n)$ for some r, n, $1 \leqq r \leqq n$, then $Tf_i = f_{i+1}$.

(b) If $i = na_n + rb_n$ then $Tf_i = \varepsilon_1 f_{i+1} - b_n \varepsilon_2 f_{i+1-b_n}$, where

$$\varepsilon_2 = 2^{(na_n + 1 - b_n/2)/\sqrt{b_n}}, \qquad \varepsilon_1 = \begin{cases} \varepsilon_2, & r < n, \\ 2^{(1+\nu_n - c_n/2)/\sqrt{c_n}}, & r = n. \end{cases}$$

So $\| Tf_i \| < 1$ for all such i, p.$\boldsymbol{d}$.

Case 4. (a) If $i \in (na_n + (r-1)b_n, r(a_n + b_n) - 1)$ for some $r, n, 0 < r < n$, then $Tf_i = 2^{1/\sqrt{b_n}} \cdot f_{i+1}$, $\| Tf_i \| = 2^{1/\sqrt{b_n}} < 1 + \eta$ for all n p.$\boldsymbol{d}$.

(b) If $i = r(a_n + b_n) - 1$ then

$$Tf_i = 2^{-(ra_n + b_n/2 - 1)/\sqrt{b_n}} \cdot \left(b_n^r \left(f_0 + \frac{1}{a_{n-r}} \cdot f_{ra_n} \right) + \sum_{j=0}^{r-1} b_n^j f_{ra_n + (r-j)b_n} \right)$$

hence $\| Tf_i \| < 1$ for all such i p.$\boldsymbol{d}$.

Case 5. (a) If for some $p \in \Lambda_n$ we have $i \in [p(c_n), p(c_n) + \nu_n)$ then $Tf_i = f_{i+1}$.

(b) If $i = p(c_n) + \nu_n$, $p \in \Lambda_n$, then

$$Tf_i = 2^{1-|p|} \cdot c_n \left(e_{1+p(c_n)+\nu_n} + \sum_{k=0}^{\nu_n} \alpha_k e_{1+p(c_n)+\nu_n - c_n^d + k} \right)$$

where $\tilde{p}_{d,\nu_n}(t) = \Sigma\, \alpha_k t^k$ and d is the degree of p. Since $| \tilde{p}_{d,\nu_n} | \leqq 1$, we know that

$$\| Tf_i \| \leqq c_n \left(\| e_{1+p(c_n)+\nu_n} \| + \max_{0 \leqq k \leqq \nu_n} \| e_{1+p(c_n) - c_n^d + \nu_n + k} \| \right). \tag{4.5.1}$$

Now (2.8) gives, for all $p \in \Lambda_n \cup \{0\}$,

$$\| e_{1+p(c_n)+\nu_n} \| = \begin{cases} 2^{(1+\xi_n - a_{n+1}/2)/\sqrt{a_{n+1}}}, & p = \hat{p}_n \\ 2^{(1+\nu_n - (p^+(c_n) - p(c_n))/2)/c_n^{J(p)-1/2}}, & p < \hat{p}_n \end{cases}$$

$$\leqq 2^{-\sqrt{c_n}/3} \quad \text{p.}\boldsymbol{d}., \quad \text{by (2.5.2)}. \tag{4.5.2}$$

Similarly, $q(t) = p(t) - t^d \in \Lambda_n \cup \{0\}$ so (2.8.6) defines f_r for each

$$r \in (q(c_n) + v_n, q^+(c_n)) \supset (1 + q(c_n) + v_n, 1 + q(c_n) + 2v_n)$$

(since $q^+(c_n) - q(c_n) \geqq c_n$ p.$\boldsymbol{d}$.). Hence, for each $0 \leqq k \leqq v_n$.

$$\| e_{1+p(c_n)-c_n^d+v_n+k} \| = 2^{((q(c_n)-q^+(c_n))/2+1+v_n+k)/c_n^{J(q)-1/2}} \tag{4.5.3}$$
$$\leqq 2^{-\sqrt{c_n}/3} \quad \text{p.}\boldsymbol{d}.$$

So by (4.5.1), (4.5.2), (4.5.3),

$$\| Tf_i \| \leqq 2c_n \cdot 2^{-\sqrt{c_n}/3} < 1 \tag{4.5.4}$$

for all n, p.$\boldsymbol{d}$.

Case 6. (a) If $i \in (p(c_n) + v_n, p^+(c_n) - 1)$, $p \in \Lambda_n \cup \{0\}$, with $p < \hat{p}_n$ then by (2.8.6), $Tf_i = 2^{1/\sqrt{c_n}} \cdot f_{i+1}$, hence

$$\| Tf_i \| = 2^{1/\sqrt{c_n}} < 1 + \eta \qquad \text{for all } n, \quad \text{p.}\boldsymbol{d}.$$

(b) If $i = p^+(c_n) - 1$ for some $p^+ \in \Lambda_n$, then by (2.8.6), (2.5.2),

$$Tf_i = 2^{(1+(p(c_n)-p^+(c_n))/2)/c_n^{J(p)-1/2}} \cdot e_{p^+(c_n)},$$

$$\| Tf_i \| \leqq 2^{(1-3c_n^J/4)/c_n^{J-1/2}} \| e_{p^+(c_n)} \| \qquad (J = J(p)) \tag{4.6.1}$$

since $p^+(c_n) - p(c_n) \geqq c_n$.

We need to estimate what is $\max_{\Lambda_n} \| e_{p(c_n)} \|$, so let us generalise slightly by writing

$$L_r = \max\{ \| e_j \| : j \in p(c_n) + [0, v_n], p \in \Lambda_n \cup \{0\}, |p| = r\}. \tag{4.6.2}$$

(2.9.3) implies that $L_0 \leqq M_2(n, b_n)$. If $d = \deg p$, $|p| = r > 0$, then for all $j \in p(c_n) + [0, v_n]$, (2.8.5) gives

$$e_j = c_n^{-1} \cdot 2^{r-1} f_j + \tilde{p}_{d,v_n}(T) e_{j-c_n^d},$$

$$\| e_j \| \leqq c_n^{-1} \cdot 2^{r-1} + \| \tilde{p}_{d,v_n}(T) e_{j-c_n^d} \|.$$

Since $|\tilde{p}_{d,v_n}| \leqq 1$ and $\deg \tilde{p}_{d,v_n} \leqq v_n$, we know that

$$\tilde{p}_{d,v_n}(T) e_{j-c_n^d} \in \Delta\{e_k, k \in [0, 2v_n] + q(c_n)\}$$

where Δ denotes the absolutely convex hull and $q(t) = p(t) - t^d$, $|q| = r - 1$. Thus for all $j \in p(c_n) + [0, v_n]$

$$\| e_j \| \leqq c_n^{-1} \cdot 2^{r-1} + \max(L_{r-1}, \max\{ \| e_k \| : k \in q(c_n) + (v_n, 2v_n]\}).$$

In view of (4.5.3), the last "max" on the right-hand side is less than 1 p.$\boldsymbol{d}$.; hence

$$L_r \leqq c_n^{-1}\cdot 2^{r-1} + \max(L_{r-1}, 1);$$

since $L_0 \leqq M_2(n, b_n)$ this gives

$$L_r \leqq M_2(n, b_n) + c_n^{-1} \sum_{s=1}^{r} 2^{s-1}$$

$$\begin{aligned} &< M_2(n, b_n)\cdot c_n^{-1}\cdot 2^r \\ (4.6.3) \qquad &\leqq M_2(n, b_n)\cdot c_n^{-1}\cdot 2^{h_n(M(v_n)+1)} \qquad \text{for } r = |p| \leqq |p|_0(1+\deg p) \\ &\leqq M_2(n, b_n)(2c_n)^{2+2M(v_n)} \end{aligned}$$

since $h_n = \lceil 2\log_2 c_n \rceil$. By (4.6.1), (4.6.2), (4.6.3)

$$\begin{aligned} \| Tf_i \| &\leqq 2\cdot 2^{-3\sqrt{c_n}/4}\cdot M_2(n, b_n)\cdot (2c_n)^{2+2M(v_n)} \\ &< 1 \qquad \text{for all } n \quad \text{p.}\boldsymbol{d}. \end{aligned}$$

These 7 cases exhaust all the possibilities, and we find that, p.$\boldsymbol{d}$., $\| Tf_i \| \leqq 1 + \eta$ for all i. This concludes the proof of Lemma 4.

§5. Behaviour of certain powers of T

LEMMA 5. *Let $\eta > 0$ be given. The following are true* p.$\boldsymbol{d}$.
For all $n \in \mathbf{N}$, $1 \leqq k \leqq M(v_n)$,

$$(5.0.1) \qquad \| T^{c_n^k}\circ (I - R_n^0) \| \leqq 1 + \eta.$$

Moreover,

$$(5.0.2) \qquad \sum_{i=1}^{\infty} \| T^{c_n^k}\circ (I - R_n^0)x_i \| < 1/a_{n+1}.$$

PROOF OF LEMMA 5. We shall show that, p.$\boldsymbol{d}$., for all such n, k we have

$$\sum_{i=1}^{\infty} \| T^{c_n^k}\circ (I - R_n^0)x_i \| < 1/a_{n+1}$$

and

$$\sup_i \| T^{c_n^k}\circ (I - R_n^0)g_i \| \leqq 1 + \eta.$$

This, with (1.1), completes the proof. As with Lemma 4, the proof of the first statement is fairly trivial. Assume that $\boldsymbol{d}$ increases sufficiently rapidly that $\| T \| \leqq 2$.

By (2.11), $R_n^0(x_i) = x_i$ ($i \leqq n$) or 0 ($i > n$). Thus, since $\| T \| \leqq 2$,

$$\sum_{i=1}^{\infty} \| T^{c_n^k} \circ (I - R_n^0) x_i \| \leqq 2^{c_n^k - 1} \sum_{i=n+1}^{\infty} \| T x_i \| \leqq 2^{c_n^k - 1} \sum_{i=n+1}^{\infty} 4 \cdot 2^{-\sqrt{a_i}/2}$$

by (4.0.1),

$$\leqq 2^{c_n^M + 1} \cdot \sum_{i=n+1}^{\infty} 2^{-\sqrt{a_i}/2}$$

where $M = M(v_n) \geqq k$,

$$\leqq 1/a_{n+1}$$

for all $n \in \mathbf{N}$, p.$\boldsymbol{d}$.

Next we show that for all i,

$$\| T^{c_n^k} \circ (I - R_n^0) g_i \| \leqq 1 + \eta,$$

equivalently,

$$\| T^{c_n^k} \circ (I - R_n^0) f_i \| \leqq 1 + \eta$$

for all i, n. As with Lemma 4, we consider several cases.

Case 0. $0 \leqq j \leqq \xi_n$. Then $(I - R_n^0) f_j = 0$ so $\| T^{c_n^k} \circ (I - R_n^0) f_j \| = 0$.

Case 1. For some $m > n$, $1 \leqq r \leqq m$, $j \in [0, \xi_{m-r}] + r a_m$. Then we consider two sub-cases: $m - n < r$ and $m - n \geqq r$.

If $m - n < r$ then by (2.11), $R_n^0(f_j) = -a_{m-r} e_{j - r a_m}$. By 2.11, then, $(I - R_n^0) f_j = a_{m-r} e_j$, and $T^{c_n^k} \circ (I - R_n^0) f_j = a_{m-r} e_{j + c_n^k}$. Since $m - r < n$ we will have

$$j + c_n^k \geqq r a_m + c_n^k > r a_m + \xi_{m-r} \qquad \text{p.}\boldsymbol{d}.,$$

so for some $0 \leqq \alpha < c_n^k$, we have

$$\| e_{j + c_n^k} \| = \| T^{\alpha}(e_{r a_m + \xi_{m-r} + 1}) \| \leqq 2^{\alpha} \cdot \varepsilon_1$$

where, as in (4.1.2),

$$\varepsilon_1 = \varepsilon_1(m, r) = \begin{cases} 2^{(1 + \xi_{m-r} - a_m/2)}/\sqrt{a_m}, & r < m, \\ 2^{(1 + \mu_m - b_m/2)}/\sqrt{b_m}, & r = m. \end{cases}$$

At any rate, $\varepsilon_1 < 2^{-\sqrt{a_m}/4}$ for all $1 \leqq r \leqq m$ p.$\boldsymbol{d}$., so

$$\| T^{c_n^k} \circ (I - R_n^0) f_j \| \leqq a_{m-r} \cdot 2^{c_n^k} \cdot 2^{-\sqrt{a_m}/4}$$

with $k \leqq M(v_n)$;

$$< 1$$

for all $m > n$ p.$\boldsymbol{d}$. If $m - n \geqq r$ then by (2.11), $R_n^0(f_j) = 0$, so

$$T^{c_n^k} \circ (I - R_n^0) f_j = T^{c_n^k} f_j = a_{m-r}(e_{j+c_n^k} - e_{j-ra_m+c_n^k}).$$

If $j + c_n^k \leqq ra_m + \xi_{m-r}$, then the right-hand side is just $f_{j+c_n^k}$; if $j + c_n^k > ra_m + \xi_{m-r}$ then for some $0 \leqq \alpha < c_n^k$, the right-hand side is

$$\alpha_{m-r} T^{\alpha}(e_{ra_m+\xi_{m-r}+1} - e_{\xi_{m-r}+1}) = a_{m-r} T^{\alpha}(\varepsilon_1 f_{ra_m+\xi_{m-r}+1} - \varepsilon_2 e_{\xi_{m-r}+1})$$

where, by (4.1.1), (4.1.2),

$$\varepsilon_2 = 2^{(1+\xi_{m-r}-a_{m-r+1}/2)/\sqrt{a_{m-r+1}}} < 2^{-\sqrt{a_{m-r+1}}/4} \quad \text{p.}\boldsymbol{d}.,$$

$$\varepsilon_1 = 2^{(1+\xi_{m-r}-a_m/2)/\sqrt{a_m}} < 2^{-\sqrt{a_m}/4} \quad \text{p.}\boldsymbol{d}. \quad (\text{since } r < m).$$

So

$$\| T^{c_n^k} f_j \| \leqq a_{m-r} \cdot 2^{\alpha}(\varepsilon_1 + \varepsilon_2) \leqq a_{m-r} \cdot 2^{c_n^M}(2^{-\sqrt{a_{m-r+1}}/4} + 2^{-\sqrt{a_m}/4})$$

$(M = M(v_n))$

$$< 1 \qquad \text{p.}\boldsymbol{d}.$$

for all m, r such that $m - r \geqq n$. So for all j covered by Case 1, we find that $\| T^{c_n^k} \circ (I - R_n^0) f_j \| \leqq 1$.

Case 2. If for some $m > n$ we have $j \in (ra_m + \xi_{m-r}, (r+1)a_m)$ (respectively, $j \in (\xi_{m-1}, a_m)$) for some $1 \leqq r < m$, then $R_n^0 f_i = 0$ by (2.11), and by (2.8.2), $f_i = 2^{(h-i)/\sqrt{a_m}} \cdot e_i$, where $h = (r + \frac{1}{2})a_m$ (respectively, $h = \frac{1}{2}a_m$), so

$$T^{c_n^k} \circ (I - R_n^0) f_i = T^{c_n^k} f_i = 2^{(h-i)/\sqrt{a_m}} \cdot e_{i+c_n^k}. \tag{5.2.1}$$

If $i + c_n^k < (r+1)a_m$ (respectively, $i + c_n^k < a_m$) then this is equal to $2^{c_n^k/\sqrt{a_m}} \cdot f_{i+c_n^k}$, and we deduce that

$$\| T^{c_n^k} \circ (I - R_n^0) f_i \| = 2^{c_n^k/\sqrt{a_m}} \leqq 1 + \eta \qquad (\text{all } m > n, k \leqq M(v_n)) \quad \text{p.}\boldsymbol{d}.$$

If $i + c_n^k \geqq (r+1)a_m$ (respectively, $i + c_n^k \geqq a_m$) then, for some $0 \leqq \alpha < c_n^k$, (5.2.1) is equal to $2^{(h-i)/\sqrt{a_m}} \cdot T^{\alpha} e_{(r+1)a_m}$. It is immediate from (2.8.1), (2.8.0) that $\| e_{(r+1)a_m} \| \leqq 2$, and then we have

$$\| T^{c_n^k} \circ (I - R_n^0) f_i \| \leqq 2^{(h-i)/\sqrt{a_m}} \cdot \| T \|^{\alpha} \cdot 2$$

$$\leqq 2^{(c_n^k - a_m/2)/\sqrt{a_m}} \cdot 2^{c_n^M} \cdot 2 \qquad (M = M(v_n))$$

since $h = (r + \frac{1}{2})a_m$ (respectively $\frac{1}{2}a_m$) and $i + c_n^k \geqq (r+1)a_m$ (respectively a_m), and $\alpha \leqq c_n^k \leqq c_n^{M(v_n)}$. Then writing $M = M(v_n)$,

$$\| T^{c_n^k} \circ (I - R_n^0) f_i \| \leqq 2^{1 + c_n^M(1+\sqrt{a_m})} \cdot 2^{-\sqrt{a_m}/2} < 1 \quad \text{for all } m > n \quad \text{p.}\boldsymbol{d}.$$

Case 3. If for some $m > n$ we have $i \in [r(a_m + b_m), ma_m + rb_m]$ then $R_m^0 f_i = 0$ by (2.11), so we consider the norm of the vector

$$T^{c_n^k} f_i = e_{i + c_n^k} - b_m e_{i + c_n^k - b_m}.$$

If $i + c_n^k \leqq ma_m + rb_m$ then, by (2.8.3), the right-hand side is $f_{i+c_n^k}$, which is of course of norm 1. If $i + c_n^k > ma_m + rb_m$ then for some α, $0 \leqq \alpha < c_n^k$, we have

$$\begin{aligned} T^{c_n^k} f_i &= T^\alpha(e_{ma_m + rb_m + 1} - b_m e_{ma_m + (r-1)b_m + 1}) \\ &= T^\alpha(\varepsilon_1 f_{ma_m + rb_m + 1} - b_m \varepsilon_2 f_{ma_m + (r-1)b_m + 1}) \end{aligned}$$

where

$$\varepsilon_2 = 2^{(ma_m + 1 - b_m/2)/\sqrt{b_m}} \quad \text{and} \quad \varepsilon_1 = \begin{cases} 2^{(1 + v_m - c_m/2)/\sqrt{c_m}}, & r = m, \\ \varepsilon_2, & r < m. \end{cases}$$

At any rate $\varepsilon_1 \vee \varepsilon_2 \leqq 2^{-\sqrt{b_m}/4}$ p.$\boldsymbol{d}$., so

$$\begin{aligned} \| T^{c_n^k} f_i \| &\leqq \| T \|^\alpha (1 + b_m) \cdot 2^{-\sqrt{b_m}/4} \\ &\leqq 2^{c_n^M}(1 + b_m) \cdot 2^{-\sqrt{b_m}/4} \qquad (M = M(v_n)) \\ &< 1 \end{aligned}$$

for all $m > n$ p.$\boldsymbol{d}$.

Case 4. If for some $m > n$, $0 \leqq r < m$ we have

$$i \in (ma_m + rb_m, (r+1)(a_m + b_m))$$

then $R_n^0 f_i = 0$, so again we consider $\| T^{c_n^k} f_i \|$; now by (2.8.4),

$$\begin{aligned} T^{c_n^k} f_i &= T^{c_n^k}(2^{(h-i)/\sqrt{b_m}} \cdot e_i) \qquad (h = (r + \tfrac{1}{2})b_m) \\ &= 2^{(h-i)/\sqrt{b_m}} \cdot e_{i + c_n^k}. \end{aligned} \tag{5.4.1}$$

If $i + c_n^k < (r+1)(a_m + b_m)$ then this is equal to $2^{c_n^k/\sqrt{b_m}} f_{i + c_n^k}$, a vector of norm $2^{c_n^k/\sqrt{b_m}} < 1 + \eta$ for all $n < m$ p.$\boldsymbol{d}$. If $i + c_n^k \geqq (r+1)(a_m + b_m)$ then for some α, $0 \leqq \alpha < c_n^k$, (5.4.1) is equal to $2^{(h-i)/\sqrt{b_m}} T^\alpha e_{(r+1)(a_m+b_m)}$ so

$$\begin{aligned}\| T^{c_n^k} f_i \| &\leqq 2^{((h-i)/\sqrt{b_m}+\alpha)} \| e_{(r+1)(a_m+b_m)} \| \\ &= 2^{((h-i)/\sqrt{b_m}+\alpha)} \left\| \left(\sum_{j=0}^{r} b_m^j f_{(r+1)a_m+(r+1-j)b_m} \right) \right. \\ &\qquad \left. + b_m^{r+1} \left(f_0 + \frac{1}{a_{m-r-1}} f_{(r+1)a_m} \right) \right\| \\ &\leqq 2^{((h-i)/\sqrt{b_m}+\alpha)} (m+2) b_m^m \\ &\leqq 2^{((c_n^k - b_m/2)/\sqrt{b_m}+M)} (m+2) b_m^m \end{aligned}$$

($M = M(v_n)$) (since $h = (r+\frac{1}{2})b_m$ and $i + c_n^k \geqq (r+1)b_m$)

$$\leqq 2^{-\sqrt{b_m}/2} (m+2) b_m^m \cdot 2^{2c_n^M} < 1$$

for all $n < m$ p.$\boldsymbol{d}$.

Case 5. If for some $m > n$, $p \in \Lambda_m$, we have $i \in [p(c_m), p(c_m) + v_m]$ then $R_n^0 f_i = 0$, and if $i + c_n^k \leqq p(c_m) + v_m$ then $T^{c_n^k} f_i$ is simply $f_{i+c_n^k}$. If, on the other hand, $i + c_n^k > p(c_m) + v_m$ then for some α, $0 \leqq \alpha < c_n^k$,

$$T^{c_n^k} f_i = T^{\alpha} \cdot T f_{p(c_m)+v_m}.$$

By (4.5.4),

$$\| T f_{p(c_m)+v_m} \| \leqq 2c_m \cdot 2^{-\sqrt{c_m}/3}$$

so

$$\begin{aligned}\| T^{c_n^k} f_i \| &\leqq 2^{1+c_n^M} c_m \cdot 2^{-\sqrt{c_m}/3} \qquad (M = M(v_n)) \\ &< 1\end{aligned}$$

for every $m > n$ p.$\boldsymbol{d}$.

Case 6. If for some $m > n$, $p \in \Lambda_m \cup \{0\}$ such that $p < \hat{p}_m$ we have $i \in (p(c_m) + v_m, = p^+(c_m))$, then $R_n^0 f_i = 0$ by (2.11), and by (2.8.6),

$$T^{c_n^k} f_i = 2^{(h-i)/c_m^{J(p)-1/2}} \cdot e_{i+c_n^k}$$

where $h = (p(c_m) + p^+(c_m))/2$. Therefore, if $i + c_n^k < p^+(c_m)$, we have

$$T^{c_n^k} f_i = 2^{c_n^k / c_m^{J(p)-1/2}} \cdot f_{i+c_n^k}$$

which, as in cases 2 and 4, is a vector of norm less than $1 + \eta$ p.$\boldsymbol{d}$. If $i + c_n^k \geqq p^+(c_m)$ then for some α, $0 \leqq \alpha < c_n^k$, we have

$$T^{c_n^k} f_i = T^{\alpha} (2^{(h-i)/c_m^{J(p)-1/2}} \cdot e_{p^+(c_m)}),$$

$$\begin{aligned} \| T^{c_n^k} f_i \| &\leqq 2^{\alpha+(h-i)/c_m^{J(p)}-1/2} \| e_{p^+(c_m)} \| \\ &\leqq (2c_m)^{2+2M(v_m)} \cdot M_2(m, b_m) \cdot 2^{\alpha+(h-i)/c_m^{J(p)}-1/2} \end{aligned}$$

by (4.6.3). Now $i + c_n^k \geqq p^+(c_m)$, $h = (p^+(c_m) + p(c_m))/2$ and by (2.5.2),

$$p^+(c_m) - p(c_m) \geqq \tfrac{3}{4} c_m^{J(p)};$$

hence,

$$\begin{aligned} \| T^{c_n^k} f_i \| &\leqq (2c_m)^{2+2M(v_m)} M_2(m, b_m) \cdot 2^{\alpha+(c_n^k - 3c_m^{J(p)}/8)/c_m^{J(p)}-1/2} \\ &\leqq (2c_m)^{2+2M(v_m)} M_2(m, b_m) \cdot 2^{2c_n^M} \cdot 2^{-3\sqrt{c_m}/8} \qquad (M = M(v_n)) \\ &< 1 \end{aligned}$$

for all $m > n$ p.$\boldsymbol{d}$.

We conclude that $\| T^{c_n^k} \circ (I - R_n^0) f_i \| \leqq 1 + \eta$ for all $n \geqq 1$, $k \leqq M(v_n)$ and $i \geqq 0$, hence $\| T^{c_n^k} \circ (I - R_n^0) g_i \| \leqq 1 + \eta$; and this concludes the proof of Lemma 5.

§6. Concerning the operators $\tau_{n,m} \circ Q_m^0$

Definition 6.1. Given $n, m \in \mathbf{N}$, $n < m$, and the underlying sequence $\boldsymbol{d}$, let $K_{n,m} \subset F_{\mu_m}$ be the set $\{x \in F_{\mu_m} : \| x \| \leqq a_m,\ \| \tau_{n,m}(x) \| \geqq 1/a_m\}$. $K_{n,m}$ is a compact subset of F_{μ_m} depending on the underlying space X, and the elements $(a_i)_{i=1}^m$, $(b_i)_{i=1}^{m-1}$, $(c_i)_{i=1}^{m-1}$ of the sequence $\boldsymbol{d}$.

Definition 6.2. Given $m \in \mathbf{N}$, let

$$\Delta_m = \{g_i\} \setminus \{f_i : 0 \leqq i \leqq \mu_m\} \subset \{g_i\}.$$

We now aim for the following lemma.

Lemma 6. P.$\boldsymbol{d}$. *the following is true.*

For all $x \in X$ *with* $\| x \| = 1$, *for all* $N \in \mathbf{N}$ *and for all* $\eta > 0$, *there are* $m, n \in \mathbf{N}$ *with* $m > n > N$ *such that*

$$Q_m^0(x) \in K_{n,m} \tag{6.3}$$

and

$$\| \pi_{\Delta_m}(x) \| < \eta. \tag{6.4}$$

Proof. First we note that the Δ_i's are nested and their intersection is empty, so for any $x \in X$ we may choose an m_0 such that for all $m \geqq m_0$,

$$\| \pi_{\Delta_m}(x) \| < \eta.$$

Given arbitrary $x \in X$ with $\| x \| = 1$, let us do exactly that, and then seek integers m, n with $m > n > N$, $m \geqq m_0$, such that (6.3) is satisfied. Without loss of generality $N = m_0$ so it is sufficient to satisfy (6.3) with $n > N$.

Since $\| x \| = 1$ we know that $\| Q_m^0 x \| \leqq a_m$ for all m by (3.2). The problem is to find an $m > n > N$ such that

$$\| \tau_{n,m} \circ Q_m^0(x) \| \geqq 1/a_m.$$

Writing $x = y + w$ ($y \in l_1$, $w \in W$) we note that by (3.1.1),

$$Q_n(x) = Q_n(y) + \sum_{i=1}^{n} x_i^*(w) x_i.$$

Now $\max_i \| Q_n f_i \| = 1$ since $Q_n f_i$ is either zero or some f_j; hence, $\| Q_n |_{l_1} \| = 1$. But $F \cap l_1$ is dense in l_1, and for all $y_0 \in F$, $Q_n(y_0) = y_0$ for all but finitely many n. Hence, $Q_n(y) \xrightarrow{n \to \infty} y$. It follows that

$$\begin{aligned} \liminf_{n \to \infty} \| Q_n(x) \| &= \| y \| + \liminf_{n \to \infty} \left\| \sum_{1}^{n} x_i^*(w) x_i \right\| \\ &\geqq \| y \| + \sup_i | x_i^*(w) | \\ &= ||| x |||, \quad \text{say} \\ &> 0. \end{aligned}$$

Let us choose an $n > N$ so large that

$$\| Q_n x \| \geqq \tfrac{1}{2} ||| x ||| \geqq 2/a_n.$$

We know that $\| Q_n P_{n,k} \| \leqq a_{n+1}$ for all $k > n$, yet for all $z \in F$, $Q_n P_{n,k} z = Q_n z$ for all but finitely many k, hence

$$Q_n P_{n,k} x \xrightarrow{k \to \infty} Q_n x.$$

Choose k so large that $\| Q_n P_{n,k} x \| \geqq 1/a_n$. Then

$$\| P_{n,k} x \| \geqq \frac{1}{a_n \cdot \| Q_n \|} \geqq \frac{1}{n a_n}.$$

Now if $\| \tau_{n,k} \circ Q_k^0 x \| > 1/2 n a_n > 1/a_k$ p.*d*., then our assertion is proved. If not, then since

$$\| P_{n,k} x \| = \| \tau_{n,k} \circ Q_k x \| \geqq 1/n a_n$$

we have

$$\| \tau_{n,k} \circ (Q_k^0 - Q_k)x \| > 1/2na_n. \tag{6.4}$$

From (2.8.1) and Definitions 2.10, 2.11 we obtain for all $j \geqq 0$,

$$(Q_k - Q_k^0) f_j$$

$$= \begin{cases} a_{m-r} e_{j - ra_m + (r-m+k)a_k}, & j \in [0, \xi_{m-r}] + ra_m, \quad 0 < m-k < r \leqq m \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\tau_{n,k} \circ (Q_k - Q_k^0) f_j$$

$$= \begin{cases} a_{m-r} e_{j - ra_m + (r-m+k)a_k}, & j \in [0, \xi_{m-r}] + ra_m, \quad 0 < m-k < r < m-n \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\tau_{n,k} \circ (Q_k - Q_k^0) = \tau_{n,k} \circ (Q_k - Q_k^0) \circ \pi_s$$

where

$$S = \left\{ f_j : j \in \bigcup_{\substack{m > k \\ r \in (m-k, m-n)}} ([0, \xi_{m-r}] + ra_m) \right\} \subset \{g_i\};$$

so

$$S = \bigcup_{m > k} S_m \qquad \text{where } S_m = \bigcup_{r \in (m-k, m-n)} ([0, \xi_{m-r}] + ra_m).$$

So if, as in (6.4),

$$\| \tau_{n,k}(Q_k - Q_k^0)x \| = \| \tau_{n,k}(Q_k - Q_k^0)\pi_s x \| > 1/2na_n$$

then

$$\| \pi_s x \| > (2na_n \| \tau_{n,k} \| (\| Q_k \| + \| Q_k^0 \|))^{-1}$$

$$\geqq (2na_n \max(k, a_{n+1}) \cdot (k + a_k))^{-1}$$

(by Lemmas 3.1, 3.2, 3.3). This implies that for some $m > k$,

$$\| \pi_{s_m} x \| > 2^{k-m} (2na_n \max(k, a_{n+1}) \cdot (k + a_k))^{-1}.$$

However, it is not hard to check from the definition that if $j \in S_m$ then

$\tau_{n,m}Q_m^0 f_j = f_j$. Moreover, for all j, $Q_m^0 f_j$ is either f_j or an element in $F_{\xi_{m-1}}$. Hence, if $j \notin S_m$, then $\pi_{S_m} \circ (\tau_{n,m} Q_m^0 f_j) = 0$. It follows that

$$\pi_{S_m} = \pi_{S_m} \circ \tau_{n,m} \circ Q_m^0.$$

Therefore, if

$$\| \pi_{S_m} x \| \geqq 2^{k-m}(2na_n \max(k, a_{n+1}) \cdot (k + a_k))^{-1}$$

then, since $\| \pi_{S_m} \| = 1$, we have

$$\begin{aligned} \| \tau_{n,m} \circ Q_m^0 x \| &\geqq 2^{k-m}(2na_n \max(k, a_{n+1}) \cdot (k + a_k))^{-1} \\ &\geqq 1/a_m, \end{aligned}$$

for all $n < k < m$, p.*d*. So there is indeed an $m > n > N$ such that $\| \tau_{n,m} \circ Q_m^0 x \| \geqq 1/a_m$. This concludes the proof of Lemma 6.

§7. Further properties of the compact set $K_{n,m}$

For the rest of this paper we assume that the operators T, Q_n, Q_n^0, etc. have been extended continuously from F to X.

Definition 7.1. For each $m \in \mathbf{N}$ let T_m denote the 'truncated' right shift operator on F_{μ_m}; T_m is the unique linear map

$$T_m : F_{\mu_m} \to F_{\mu_m} : e_i \to \begin{cases} e_{i+1}, & 0 \leqq i < \mu_m, \\ 0, & i = \mu_m. \end{cases}$$

Lemma 7.2. *There is a function $N_1 : \mathbf{N}^2 \to \mathbf{N}$ with the following property. P.d., for all $n < m$ and $x \in K_{n,m}$ there is a polynomial p with*

(7.2.1) $$|p| \leqq N_1(m, a_m),$$

(7.2.2) $$\deg p \leqq ma_m, \qquad t^{a_m} \mid p(t),$$

and

(7.2.3) $$\| p(T_m)x - e_0 \| \leqq \frac{1}{a_m} + \frac{1}{a_{n-1}}.$$

Proof. Given $x \in K_{n,m}$, write $x = \Sigma_{i=\alpha}^{\mu_m} \lambda_i e_i$, where $\lambda_\alpha \neq 0$. Then

$$\operatorname{lin}\{T_m^r x : a_m \leqq r \leqq \mu_m\} = \operatorname{lin}\{e_{\alpha+a_m}, e_{\alpha+a_m+1}, \ldots, e_{\mu_m}\}.$$

In particular, since $\tau_{n,m}(x) \neq 0$, we have $\alpha < (m-n)a_m$ so certainly

$$e_{(m-n+1)a_m} \in \operatorname{lin}\{T_m^r x : a_m \leqq r \leqq \mu_m\}.$$

Since $K_{n,m}$ is compact, there are a finite number of polynomials $p_1, p_2, \ldots, p_k$,

$$p_j(t) = \sum_{i=a_m}^{ma_m} \lambda_{ij} t^i$$

such that for all $x \in K_{n,m}$ there is a j for which

$$\| p_j(T_m)x - e_{(m-n+1)a_m} \| < 1/a_m.$$

Write $N = \max_j | p_j |$; and note that we can write $N \leqq N_1(m, a_m)$ for a suitable function $N_1 : \mathbf{N}^2 \to \mathbf{N}$ (since things depend only on the choices of n, m, $a_1, \ldots, a_m$, $b_1, \ldots, b_{m-1}$, $c_1, \ldots, c_{m-1}$). Since $\| e_{(m-n+1)a_m} - e_0 \| a_{n-1}^{-1}$, we have the result.

§8. More about the operators $T^{c_n^k}$

In this section we aim for the following lemma.

LEMMA 8. P.*d*. *the following is true.*
For all $n \in \mathbf{N}$, $1 \leqq k \leqq M(\nu_n)$,

$$\| T^{c_n^k} \circ \pi_H \| < 6,$$

where $H = \{ f_i : i \in (\nu_n, \xi_n] \}$.

DEFINITION 8.1. For each $n \in \mathbf{N}$, $r \in \mathbf{Z}^+$ with $0 \leqq r \leqq h_n \cdot M(\nu_n)$, let

$$L_r^{(n)} = \max \left\{ \left\| e_{i+p(c_n)} - \prod_{j=1}^{M(\nu_n)} (\tilde{p}_{j,\nu_n}(T))^{\alpha_j} e_i \right\| : \right.$$

(8.1.1)
$$\left. p \in \Lambda_n \cup \{0\}, \, | p | = r, \, p(t) = \sum_{j=0}^{M(\nu_n)} \alpha_j t^j, \, i \in [0, \nu_n] \right\}$$

and let

$$\tilde{L}_r^{(n)} = \max \left\{ \left\| e_{i+p(c_n)} - \prod_{j=1}^{M(\nu_n)} (\tilde{p}_{j,\nu_n}(T))^{\alpha_j} e_i \right\| : \right.$$

$$\left. p \in \Lambda_n \cup \{0\}, \, | p | = r, \, p(t) = \sum_{j=1}^{M(\nu_n)} \alpha_j t^j, \, i \in [0, 2\nu_n] \right\}.$$

LEMMA 8.2. P.*d*., $\tilde{L}_r^{(n)} \leqq \max(L_r^{(n)}, c_n^{-1})$ *for all* $n \in \mathbf{N}$, $0 \leqq r \leqq h_n M(\nu_n)$.

PROOF. Let $i \in (\nu_n, 2\nu_n]$, and let $| p | = r$, $p \in \Lambda_n \cup \{0\}$. By (4.5.2), for all $q \in \Lambda_n \cup \{0\}$ we have

$$\| e_{1+q(c_n)+v_n} \| \leqq 2^{-\sqrt{c_n}/3};$$

in particular, using the fact that $\| T \| \leqq 2$ p.*d*.,

$$\| e_{i+p(c_n)} \| \leqq 2^{i-v_n-\sqrt{c_n}/3} \leqq 1/2c_n \qquad \text{p.}\boldsymbol{d}., \tag{8.2.1}$$

and writing $p(t) = \Sigma_0^{M(v_n)} \alpha_j t^j$,

$$\left\| \prod_1^{M(v_n)} (\bar{p}_{j,v_n}(T))^{\alpha_j} e_i \right\| \leqq 2^{\deg \bar{p}} \cdot \| e_i \|$$

(where $\bar{p}(t) = \Pi_1^{M(v_n)}(\bar{p}_{j,v_n}(T))^{\alpha_j}$)

$$\leqq 2^{\deg \bar{p} + i - v_n - c_n/3}.$$

But

$$\deg \bar{p} \leqq \left(\sum_j \alpha_j \right) \cdot \left(\max_i \deg \bar{p}_{j,v_n} \right) \leqq v_n \left(\sum_j \alpha_j \right) = v_n | p |$$

$$= r v_n \leqq h_n v_n M(v_n).$$

So

$$\left\| \prod_{j=1}^{M(v_n)} (\bar{p}_{j,v_n}(T))^{\alpha_j} e_i \right\| \leqq 2^{h_n v_n M(v_n) + i - v_n - \sqrt{c_n}/3}$$

$$\leqq (2c_n^2)^{v_n M(v_n)} \cdot 2^{v_n} \cdot 2^{-\sqrt{c_n}/3}$$

(since $h_n = \lceil 2 \log_2 c_n \rceil$ and $i \leqq 2v_n$)

$$\leqq \frac{1}{2c_n} \qquad \text{p.}\boldsymbol{d}. \tag{8.2.2}$$

Using (8.2.1), (8.2.2) and the triangle inequality we have, for $i \in (v_n, \ 2v_n]$,

$$\left\| e_{i+p(c_n)} - \prod_{j=0}^{M(v_n)} (\bar{p}_{j,v_n}(T))^{\alpha_j} e_i \right\| \leqq 1/c_n,$$

which implies that $\bar{L}_r^{(n)} \leqq \max(1/c_n, L_r^{(n)})$, by Definition 8.1.

LEMMA 8.3. P.*d*. *the following is true. For all* n, r, *such that* $0 \leqq r \leqq h_n M(v_n)$, *we have* $\bar{L}_r^{(n)} \leqq (2^r - 1)/c_n$.

PROOF. It is obvious that $L_0^{(n)} = \bar{L}_0^{(n)} = 0$. For $r > 0$ we get an induction argument as follows. Let $p \in \Lambda_n$, $|p| = r$ and let $d = \deg p$. By (2.8.5), for all $i \in [0, v_n]$ we have

(8.3.1) $$e_{i+p(c_n)} = \bar{p}_{d,v_n}(T)e_{i+p(c_n)-c_n^d} + \frac{2^{|p|-1}}{c_n} f_{i+p(c_n)}.$$

Write $q(t) = p(t) - t^d \in \Lambda_n \cup \{0\}$ (with $|q| = r-1$), and let

$$\bar{p}_{d,v_n}(t) = \sum_0^{v_n} \lambda_j t^j, \qquad \sum_j |\lambda_j| = |\bar{p}_{d,v_n}| \leqq 1.$$

Then

(8.3.2) $$\bar{p}_{d,v_n}(T)e_{i+p(c_n)-c_n^d} = \sum_{j=0}^{v_n} \lambda_j e_{i+q(c_n)+j}.$$

By definition of $\bar{L}_{r-1}^{(n)}$, for all $0 \leqq i+j \leqq 2v_n$ we have

(8.3.3) $$\left\| e_{i+q(c_n)+j} - \prod_{k=1}^{M(v_n)} (\bar{p}_{k,v_n}(T))^{\alpha_k} e_{i+j} \right\| \leqq \bar{L}_{r-1}^{(n)}.$$

Substituting (8.3.3) in (8.3.2) and using the triangle inequality we have

$$\left\| \bar{p}_{d,v_n}(T)e_{i+p(c_n)-c_n^d} - \sum_{j=0}^{v_n} \lambda_j \prod_{k=1}^{M(v_n)} (\bar{p}_{k,v_n}(T))^{\alpha_k} e_{i+j} \right\| \leqq \bar{L}_{r-1}^{(n)} \sum |\lambda_j| \leqq \bar{L}_{r-1}^{(n)}.$$

Writing $p(t) = \Sigma_k \bar{\alpha}_k t^k$ ($\bar{\alpha}_k = \alpha_k$ ($k \neq d$), $\alpha_k + 1$ ($k = d$)), this is

(8.3.4) $$\left\| \bar{p}_{d,v_n}(T)e_{i+p(c_n)-c_n^d} - \prod_{k=1}^{M(v_n)} (\bar{p}_{k,v_n}(T))^{\bar{\alpha}_k} e_i \right\| \leqq \bar{L}_{r-1}^{(n)}.$$

Substituting (8.3.4) in (8.3.1) and using the fact that every f_j has norm 1, we have

$$\left\| e_{i+p(c_n)} - \prod_{k=1}^{M(v_n)} (\bar{p}_{k,v_n}(T))^{\bar{\alpha}_k} e_i \right\| \leqq \bar{L}_{r-1}^{(n)} + \frac{2^{r-1}}{c_n},$$

hence

$$L_r^{(n)} \leqq \bar{L}_{r-1}^{(n)} + \frac{2^{r-1}}{c_n},$$

so if $\bar{L}_{r-1}^{(n)} \leqq (2^{r-1}-1)/c_n$ then

$$L_r^{(n)} \leqq c_n^{-1}(2^{r-1} + 2^{r-1} - 1) = \frac{2^r - 1}{c_n};$$

therefore, by Lemma 8.2,

$$\bar{L}_r^{(n)} \leqq \max\left(L_r^{(n)}, \frac{1}{c_n}\right) = \frac{2^r - 1}{c_n}.$$

So by induction on r,

$$L_r^{(n)} \leqq \frac{2^r - 1}{c_n} \qquad \text{for all } 0 \leqq r \leqq h_n M(v_n).$$

We need two more preliminary lemmas before giving the proof of Lemma 8.

LEMMA 8.4. P.*d*. *the following is true. For every* $p \in \Lambda_n \cup \{0\}$ *and every* $q(t) = p(t) + t^k$ $(1 \leqq k \leqq M(v_n))$ *such that* $q \notin \Lambda_n$, *we have*

$$\| e_{i+c_n^k} \| \leqq 2 \cdot 2^{-\sqrt{c_n}/2}$$

for all $i \in [p(c_n), p(c_n) + 2c_n^k]$.

PROOF. If $q(t) = t^k + p(t) \notin \Lambda_n$, this means (Definition 2.3) that $p(t)$ has a coefficient of h_n in t^k. Write

$$p(t) = \sum_{j=1}^{M(v_n)} \alpha_j t^j = \sum_{j=1}^{k-1} \alpha_j t^j + \sum_{j=k}^{k_1} h_n t^j + \sum_{j=k_1+1}^{M(v_n)} \alpha_j t^j \tag{8.4.1}$$

where $k_1 \geqq k$, and if $k_1 < M(v_n)$, $\alpha_{1+k_1} < h_n$.

Case 1. $k_1 < M(v_n)$. Then

$$i + c_n^k \geqq p(c_n) + c_n^k > p(c_n) + \sum_{j=1}^{k-1} h_n c_n^j + v_n \qquad \text{p.}\boldsymbol{d}.$$

(since $h_n = \lceil 2 \log_2 c_n \rceil$)

$$\geqq \sum_{j=1}^{k_1} h_n c_n^j + \sum_{j=1+k_1}^{M(v_n)} \alpha_j c_n^j + v_n = q_0(c_n) + v_n. \tag{8.4.2}$$

On the other hand,

$$\begin{aligned} i + c_n^k &\leqq p(c_n) + 3c_n^k < p(c_n) + c_n^{1+k_1} - \sum_{j=1}^{k_1} h_n c_n^j \qquad \text{p.}\boldsymbol{d}. \\ &\leqq c_n^{1+k_1} + \sum_{j=1+k_1}^{M(v_n)} \alpha_j c_n^j = q_1(c_n), \qquad \text{say.} \end{aligned} \tag{8.4.3}$$

Now the polynomials q_0 and q_1 are both in Λ_n, and moreover $q_1 = q_0^+$, $J(q_0) = 1 + k_1$. So $i + c_n^k \in (q_0(c_n) + v_n, q_1(c_n))$ and by (2.8.6),

$$e_{i+c_n^k} = \varepsilon f_{i+c_n^k}$$

where

$$\varepsilon = \| e_{i+c_n^k} \| = 2^{(i+c_n^k-h)/c_n^{1/2+k_1}}, \qquad h = (q_1(c_n) + q_0(c_n)). \tag{8.4.4}$$

Now

$$
\begin{aligned}
i + c_n^k - h &\leqq p(c_n) + 3c_n^k - h \\
&= \sum_{j=1}^{k_1} (\alpha_j - \tfrac{1}{2}h_n)c_n^j + 3c_n^k - \tfrac{1}{2}c_n^{1+k_1} \\
&\leqq -\tfrac{1}{2}c_n^{1+k_1} + c_n^{1/2+k_1} \qquad \text{p.}\boldsymbol{d}.
\end{aligned}
$$

By (8.4.4) then,

$$\varepsilon \leqq 2 \cdot 2^{-\sqrt{c_n}/2}.$$

Case 2. $k_1 = M(v_n)$. Then, p.$\boldsymbol{d}$., $i + c_n^k > \xi_n$ hence

$$\| e_{i+c_n^k} \| = \| T^\alpha e_{1+\xi_n} \|$$

(with $0 \leqq \alpha < 3c_n^k$ since $p(c_n) \leqq \xi_n$)

$$
\begin{aligned}
&\leqq 2^{3c_n^M} \| e_{1+\xi_n} \| \qquad (M = M(v_n)) \\
&= 2^{3c_n^M + (1+\xi_n - (a_{n+1})/2/\sqrt{a_{n+1}}} \qquad \text{(by (2.8.2))} \\
&< 2^{-c_n} \qquad \text{p.}\boldsymbol{d}.
\end{aligned}
$$

□

LEMMA 8.5. (a) *For all* $p \in \Lambda_n \cup \{0\}$, $p < \hat{p}_n$, *we have*

$$p^+(c_n) - p(c_n) = c_n^{J(p)} - \sum_{j=1}^{J(p)-1} h_n c_n^j, \tag{8.5.1}$$

(b) P.$\boldsymbol{d}$. *the following is true. For all* i, $0 \leqq i \leqq 2\xi_n$ *we have*

$$\| e_i \| \leqq 2^{\sqrt{c_n}/2}.$$

PROOF. (a) If $p < \hat{p}_n$ write

$$p(t) = \sum_{m=1}^{k} h_n t^m + \sum_{m=1+k}^{M(v_n)} \alpha_m t^m \qquad \text{with } k < M(v_n) \quad \text{and} \quad \alpha_{1+k} < h_n.$$

Then

$$p^+(t) = t^{1+k} + \sum_{m=1+k}^{M(v_n)} \alpha_m t^m$$

and $J(p) = 1 + k$, so (8.5.1) holds.

(b) If $0 \leqq i \leqq v_n$ then $\| e_i \| \leqq M_2(n, b_n) < 2^{\sqrt{c_n}/2}$ p.$\boldsymbol{d}$. If $i \in [p(c_n), p(c_n) + v_n]$ for some $p \in \Lambda_n$ then by (4.6.3)

$$\| e_i \| \leqq M_2(n, b_n) \cdot (2c_n)^{2+2M(v_n)} < 2^{\sqrt{c_n}/2} \qquad \text{p.}\boldsymbol{d}.$$

If $i\in(p(c_n)+\nu_n,\ p^+(c_n))$ for some $p\in\Lambda_n\cup\{0\}$, then by (2.8.6)

$$\begin{aligned}\|e_i\| &= 2^{(i-h)/c_n^{J(p)-1/2}} \qquad (h=(p(c_n)+p^+(c_n))/2)\\ &\leqq 2^{(p^+(c_n)-1-(p(c_n)+p^+(c_n))/2)/c_n^{J(p)-1/2}}\\ &\leqq 2^{(c_n^{J(p)}/2)/c_n^{J(p)-1/2}} \qquad \text{by part (a)}\\ &\leqq 2^{\sqrt{c_n}/2}.\end{aligned}$$

If $\xi_n<i\leqq 2\xi_n$ then, by (2.8.2),

$$\|e_i\| = 2^{(i-a_{n+1}/2)/\sqrt{a_{n+1}}}<1 \qquad \text{p.}d.$$

Thus Lemma 8.5 is proved.

Proof of Lemma 8. Let $H=\{f_i:\nu_n<i\leqq\xi_n\}$. Our assertion is that

(8.5) $$\|T^{c_n^k}\circ\pi_H\|<6 \qquad (n\in\mathbf{N},\ 1\leqq k\leqq M(\nu_n)).$$

π_H is a norm 1 projection, and $\operatorname{Im}\pi_H\cong l_1^{|H|}$ with basis $\{f_i:\nu_n<i\leqq\xi_n\}$, so (8.5) is equivalent to

(8.6) $$\max_{\substack{i\in(\nu_n,\xi_n]\\ 1\leqq k\leqq M(\nu_n)}}\|T^{c_n^k}f_i\|<6.$$

This inequality we prove case by case for different values of i.

Case 1. For some $p\in\Lambda_n$ we have $i\in[p(c_n),\ p(c_n)+\nu_n]$. Let us say that $|p|=r$. By (2.8.5), writing $d=\deg p$ we have

(8.7.1) $$f_i=2^{1-r}c_n(e_i-\bar p_{d,\nu_n}(T)e_{i-c_n^d})$$

and so

(8.7.2) $$T^{c_n^k}f_i=2^{1-r}c_n(e_{i+c_n^k}-\bar p_{d,\nu_n}(T)e_{i-c_n^d+c_n^k}).$$

Write $p(t)=\sum_{j=1}^{M(\nu_n)}\alpha_j t^j$.

Case 1*a*. $p(t)+t^k\in\Lambda_n$. Then by Definition 8.1, writing $\bar\alpha_j=\alpha_j\ (j\neq k)$, $\alpha_j+1\ (j=k)$ we have

(8.7.3) $$\left\|e_{i+c_n^k}-\prod_{j=1}^{M(\nu_n)}(\bar p_{j,\nu_n}(T))^{\bar\alpha_j}e_{i-p(c_n)}\right\|\leqq L_{r+1}^{(n)}.$$

Moreover, writing $\bar{\bar\alpha}_j=\bar\alpha_j\ (j\neq d)$, $\bar\alpha_j-1\ (j=d)$, we also have for all $0\leqq l\leqq\nu_n$,

$$\left\| e_{i-c_n^d+c_n^k+l} - \prod_{j=1}^{M(v_n)} (\bar{p}_{j,v_n}(T))^{\bar{\alpha}_j} e_{i-p(c_n)+l} \right\| \leqq \bar{L}_r^{(n)},$$

hence

$$\left\| \bar{p}_{d,v_n}(T) e_{i-c_n^d+c_n^k} - \prod_{j=1}^{M(v_n)} (\bar{p}_{j,v_n}(T))^{\bar{\alpha}_j} e_{i-p(c_n)} \right\| \leqq \bar{L}_r^{(n)} \, | \bar{p}_{d,v_n} | \leqq \bar{L}_r^{(n)}. \tag{8.7.4}$$

By (8.7.2), (8.7.3), (8.7.4) and the triangle inequality we have

$$\begin{aligned} \| T^{c_n^k} f_i \| &\leqq 2^{1-r} c_n (\bar{L}_r^{(n)} + L_{r+1}^{(n)}) \\ &\leqq 2^{1-r} c_n \left(\frac{2^r - 1}{c_n} + \frac{2^{r+1} - 1}{c_n} \right) \qquad \text{(by Lemma 8.3)} \\ &< 6. \end{aligned}$$

Case 1*b*. $p(t) + t^k \notin \Lambda_n$. Since $v_n < c_n$ we may apply Lemma 8.4 and obtain

$$\| e_{i+c_n^k} \| \leqq 2 \cdot 2^{-\sqrt{c_n}/2}; \tag{8.7.5}$$

if $p(t) - t^d + t^k \notin \Lambda_n$ then we similarly have

$$\| \bar{p}_{d,v_n}(T) e_{i-c_n^d+c_n^k} \| \leqq 2 \cdot 2^{-\sqrt{c_n}/2} \cdot | \bar{p}_{d,v_n} | \leqq 2 \cdot 2^{-\sqrt{c_n}/2};$$

hence, by (8.7.2),

$$\| T^{c_n^k} f_i \| \leqq 2^{1-r} c_n \cdot 2 \cdot 2 \cdot 2^{-\sqrt{c_n}/2} \leqq 4 c_n \cdot 2^{-\sqrt{c_n}/2} < 1 \qquad \text{p.}\boldsymbol{d}.$$

If $p(t) - t^d + t^k \in \Lambda_n$ this means that $d = k$ and, since $p(t) + t^k \notin \Lambda_n$, we must have $\alpha_k = h_n$, in particular

$$r \geqq h_n. \tag{8.7.6}$$

By Definition 8.1,

$$\begin{aligned} &\left\| \bar{p}_{d,v_n}(T) e_{i-c_n^d+c_n^k} - \prod_{j=1}^{M(v_n)} (\bar{p}_{j,v_n}(T))^{\bar{\alpha}_j} e_{i-p(c_n)} \right\| \\ &= \left\| \bar{p}_{d,v_n}(T) \left(e_i - \prod_{j=1}^{M(v_n)} (\bar{p}_{j,v_n}(T))^{\alpha_j} e_{i-p(c_n)} \right) \right\| \\ &\leqq | \bar{p}_{d,v_n} | \cdot \bar{L}_r^{(n)} \\ &\leqq \bar{L}_r^{(n)}; \end{aligned} \tag{8.7.7}$$

moreover

$$\left\| \prod_{j=1}^{M(\nu_n)} (\bar{p}_{j,\nu_n}(T))^{\bar{\alpha}_j} e_{i-p(c_n)} \right\| \leqq \max_{0 \leqq j \leqq \nu_n + \deg \bar{p}} \| e_j \| \tag{8.7.8}$$

(where $\bar{p} = \Pi_{j=1}^{M(\nu_n)} (\bar{p}_{j,\nu_n})^{\bar{\alpha}_j}$), since $0 \leqq i - p(c_n) \leqq \nu_n$. Now each $\deg \bar{p}_{j,\nu_n} \leqq \nu_n$ so

$$\deg \bar{p} \leqq M(\nu_n) \cdot \nu_n \cdot \max \bar{\alpha}_j \leqq M(\nu_n) \cdot \nu_n \cdot (1 + h_n) \leqq \tfrac{1}{4} c_n \qquad \text{p.}\boldsymbol{d}.,$$

since $p(t) \in \Lambda_n$. So

$$\begin{aligned} \max_{0 \leqq j \leqq \nu_n + \deg \bar{p}} \| e_j \| &\leqq \left(\max_{0 \leqq j \leqq \nu_n} \| e_j \| \right) \vee \left(\max_{\nu_n < j \leqq c_n/4} \| e_j \| \right) \\ &\leqq M_2(n, b_n) \vee 2^{-\sqrt{c_n}/4} \qquad \text{(by (2.9.2), (2.8.6))} \qquad (8.7.9) \\ &= M_2(n, b_n). \end{aligned}$$

Substitute (8.7.9) in (8.7.8), then use (8.7.7) and the triangle inequality, and we obtain

$$\| \bar{p}_{d,\nu_n}(T) e_{i - c_n^d + c_n^k} \| \leqq L_r^{(n)} + M_2(n, b_n). \tag{8.7.10}$$

Using (8.7.5) and (8.7.10) in (8.7.2) we have

$$\begin{aligned} \| T^{c_n^k} f_i \| &\leqq 2^{1-r} \cdot c_n (2 \cdot 2^{-\sqrt{c_n}/2} + L_r^{(n)} + M_2(n, b_n)) \\ &\leqq 2 + 2^{1-r} \cdot c_n (2 + M_2(n, b_n)) \qquad \text{by Lemma 8.3} \\ &\leqq 2 + 2^{1-h_n} \cdot c_n (2 + M_2(n, b_n)) \end{aligned}$$

by (8.7.6); however $h_n = \lceil 2 \log_2 c_n \rceil$ so

$$\begin{aligned} \text{R.H.S.} &\leqq 2 + c_n^{-1}(2 + M_2(n, b_n)) \\ &\leqq 3 \qquad \text{p.}\boldsymbol{d}. \end{aligned}$$

This concludes Case 1, $\| T^{c_n^k} f_i \|$ being less than 6 in all subcases.

Case 2. For some $p \in \Lambda_n \cup \{0\}$, $p < \hat{p}_n$, we have

$$i \in (p(c_n) + \nu_n, p^+(c_n)).$$

Writing

$$p(t) = \sum_j \alpha_j t^j, \quad p^+(t) = \sum_j \alpha_j^+ t^j, \quad \text{and} \quad J = J(p) = \max\{j : \alpha_j \neq \alpha_j^+\},$$

we know by (2.8.6) that

$$T^{c_n^k} f_i = 2^{(h-i)/c_n^{J(p)-1/2}} \cdot e_{i + c_n^k} \qquad (h = (p^+(c_n) + p(c_n))/2). \tag{8.8.1}$$

Note that in view of Lemma 8.5(a), the coefficient of $e_{i+c_n^k}$ is at most $2^{\sqrt{c_n}/2}$ so

$$\| T^{c_n^k} f_i \| \leqq 2^{\sqrt{c_n}/2} \| e_{i+c_n^k} \| . \tag{8.8.2}$$

Case 2a. $J(p) > k$. If $i + c_n^k \geqq p^+(c_n)$ then $i \geqq p^+(c_n) - c_n^k$ so, by (8.5.1),

$$i - h \geqq \frac{1}{2}\left(c_n^J - \sum_{j=1}^{J-1} h_n c_n^j\right) - c_n^k$$

hence

$$(i-h)/c_n^{J-1/2} \geqq \tfrac{1}{2}\sqrt{c_n} - 1 \qquad \text{p.}\boldsymbol{d}.$$

Substituting in (8.8.1) we have

$$\| T^{c_n^k} f_i \| \leqq 2^{1-\sqrt{c_n}/2} \| e_{i+c_n^k} \| \leqq 2$$

by Lemma 8.5(b). If $i + c_n^k < p^+(c_n)$ then (2.8.6) gives simply

$$T^{c_n^k} f_i = 2^{c_n^k / c_n^{J-1/2}} \cdot f_{i+c_n^k}$$

so

$$\| T^{c_n^k} f_i \| \leqq 2^{1/\sqrt{c_n}} < 2.$$

Case 2b. $J(p) \leqq k$, and $p(t) + t^k \notin \Lambda_n$. Then by Lemma 8.5(a), $i \leqq p^+(c_n) \leqq p(c_n) + c_n^k$ so $i \in [p(c_n), p(c_n) + 2c_n^k]$. By Lemma 8.4,

$$\| e_{i+c_n^k} \| \leqq 2 \cdot 2^{-\sqrt{c_n}/2}$$

so by (8.8.2) $\| T^{c_n^k} f_i \| \leqq 2$.

Case 2c. $J(p) \leqq k$, and $p(t) + t^k \in \Lambda_n$. If $J(p) \neq k$, or if the coefficient of t^k in $p^+(t)$ is less than h_n, this means that $p^+(t) + t^k \in \Lambda_n$ and

$$p^+(t) + t^k = (p(t) + t^k)^+;$$

hence by (2.8.6), $T^{c_n^k} f_i = f_{i+c_n^k}$, a vector of norm 1.

If $J(p) = k$ and the coefficient of t^k in $p^+(t)$ is h_n, then writing $q(t) = t^k + p(t)$, we either have $q = \hat{p}_n$ or $J(q) \geqq k+1$. Moreover, p.$\boldsymbol{d}$. we have

$$i + c_n^k \in (q(c_n) + v_n, p^+(c_n) + c_n^k) \subset (q(c_n) + v_n, q(c_n) + 2c_n^k)$$

by Lemma 8.5(a). If $q = \hat{p}_n$ then p.$\boldsymbol{d}$. (2.8.2) gives

$$\| e_{i+c_n^k} \| = 2^{(i - a_{n+1}/2)/\sqrt{a_{n+1}}} \leqq 2^{(\xi_n + 2c_n^k - a_{n+1}/2)/\sqrt{a_{n+1}}},$$

hence by (8.8.2)

$$\| T^{c_n^k} f_i \| \leqq 2^{(\xi_n + 2c_n^k - a_{n+1}/2)/\sqrt{a_{n+1}}} \cdot 2^{\sqrt{c_n}/2} < 1 \qquad \text{p.}\boldsymbol{d}.$$

If $q < \hat{p}_n$ then, by (2.8.6),

$$\begin{aligned} \| e_{i+c_n^k} \| &= 2^{(i-h)/c_n^{J(q)-1/2}} \qquad (h = (q(c_n) + q^+(c_n))/2) \\ &\leqq 2^{((q(c_n))/2 + 2c_n^k - (q^+(c_n))/2)/c_n J(q) - 1/2} \\ &\leqq 2 \cdot 2^{-\sqrt{c_n}/2} \qquad p.\boldsymbol{d}., \end{aligned}$$

since $k < J(q)$. (8.8.2) then gives $\| T^{c_n^k} f_i \| \leqq 4$.

In all cases we conclude that $\| T^{c_n^k} f_i \| < 6$. Thus Lemma 8 is proved.

We conclude §8 with one further remark.

LEMMA 8.9. *Let $m \in \mathbf{N}$, and let $H = \{ f_i : \nu_m < i \leqq \xi_m \}$ (as in Lemma 8). Then $\pi_H = \pi_H \circ R_m^0$.*

PROOF. If $j \in (\nu_m, \xi_m]$ then, by (2.11), $R_m^0 f_j = f_j$ so

$$\pi_H f_j = \pi_H R_m^0 f_j = f_j.$$

For $j \notin (\nu_m, \xi_m]$, (2.11) gives either $R_m^0 f_j = f_j$, or $R_m^0 f_j = 0$, or else

$$R_m^0 f_j = - a_{n-r} e_{j - ra_n}$$

when $j \in [0, \xi_{n-r}] + ra_n$ with $0 < n - m < r \leqq n$. In this last case,

$$R_m^0 f_j \in F_{\xi_{m-1}}$$

because $n - r < m$ and $j - ra_n \in [0, \xi_{n-r}]$. Hence for all $j \notin (\nu_m, \xi_m]$, we have $R_m^0 f_j \in \ker \pi_H$, so

$$\pi_H f_j = \pi_H R_m^0 f_j = 0.$$

Thus, $\pi_H = \pi_H \circ R_m^0$.

§9. The main lemma

In this section we prove a lemma which brings us very close to our main result, that T is hypercyclic.

LEMMA 9. P.$\boldsymbol{d}$. *the following is true.*

For all $m, n \in \mathbf{N}$, $2 \leqq n < m$ (9.1), and all $x \in X$, $\| x \| = 1$ (9.2), such that

$$Q_m^0(x) \in K_{n,m}, \tag{9.3}$$

and for all polynomials s with $\deg s \leqq n$, $|s| \leqq n$ (9.3.1), there is a k, $1 \leqq k \leqq M(\nu_n)$ (9.3.2), such that

(9.4) $$\| T^{c_m^k}x - s(T)e_0 \| \leqq 8 \| \pi_{\Delta_m}(x) \| + 2a_{n-1}^{-1/2}.$$

PROOF. Write

(9.4.1) $$Z = Q_m^0(x) = \sum_0^{\mu_m} z_i e_i .$$

By (7.2) there is a polynomial p with

(9.5) $$|p| \leqq N_1(m, a_m),$$

(9.6) $$\deg p \leqq \mu_m,$$

(9.7) $$t^{a_m} \mid p(t)$$

and

(9.8) $$\| p(T_m)z - e_0 \| \leqq \frac{1}{a_m} + \frac{1}{a_{n-1}} .$$

Consider

(9.9) $$q(t) = \frac{t^{b_m}}{b_m} p(t)s(t).$$

Then

(9.10) $$|q| \leqq \frac{1}{b_m} \cdot N_1(m, a_m) \cdot n < b_m^{-1/2} \qquad \text{p.}\boldsymbol{d}.,$$

and

(9.11) $$\deg q \leqq b_m + ma_m + n < \nu_m \qquad \text{p.}\boldsymbol{d}.$$

By the definition of Π_ν, (2.1), there is a k, $1 \leqq k \leqq M(\nu_m)$ such that

(9.12) $$|\bar{p}_{k,\nu_m} - q| \leqq 4^{-\nu_m}.$$

This is the k which will fit in (9.4). Now, let $H = \{ f_i : i \in (\nu_m, \xi_m] \}$ as in §8. Since by Lemma 8.9, $\pi_H = \pi_H \circ R_m^0$, we have

$$T^{c_m^k}x = T^{c_m^k}(I - R_m^0)x + T^{c_m^k}\pi_H x + T^{c_m^k}(I - \pi_H)R_m^0 x$$
$$= T^{c_m^k}(I - R_m^0)x + T^{c_m^k}\pi_H x + T^{c_m^k}(Q_m^0 x + (\pi_S - \pi_H)x)$$

since $Q_m^0 = (I - \pi_S)R_m^0$ by Definition 2.11, where $S = \{ f_i : i \in (\mu_m, \xi_m] \}$ (and $\pi_S = \pi_S \circ R_m^0$ by the argument of Lemma 8.9).

$$= T^{c_m^k}(I - R_m^0)x + T^{c_m^k}\pi_H x + T^{c_m^k}(z + (\pi_S - \pi_H)x).$$

Hence,

$$\begin{aligned}T^{c_m^k}x - s(T)e_0 = T^{c_m^k}(I - R_m^0)x + T^{c_m^k}\pi_H x + (T^{c_m^k} - \tilde{p}_{k,\nu_m}(T))(z + (\pi_S - \pi_H)x) \\ + (\tilde{p}_{k,\nu_m}(T) - q(T))(z + (\pi_S - \pi_H)x) \\ + q(T)(\pi_S - \pi_H)x + (q(T) - p(T)s(T))z \\ + (s(T)p(T) - s(T)p(T_m))z + (s(T)p(T_m)z - s(T)e_0).\end{aligned}$$

Therefore

$$\begin{aligned}\| T^{c_m^k}x - s(T)e_0 \| \leqq & \; \| T^{c_m^k}(I - R_m^0)x \| + \| T^{c_m^k}\pi_H x \| \\ & + \| (T^{c_m^k} - \tilde{p}_{k,\nu_m}(T))(z + (\pi_S - \pi_H)x) \| \\ & + \| (\tilde{p}_{k,\nu_m}(T) - q(T))(z + (\pi_S - \pi_H)x) \| \\ & + \| q(T)(\pi_S - \pi_H)x \| \\ & + \| (q(T) - p(T)s(T))z \| \\ & + \| (s(T)p(T) - s(T)p(T_m))z \| \\ & + \| s(T)p(T_m)z - s(T)e_0 \| .\end{aligned} \tag{9.13}$$

Let us estimate the 8 terms on the right-hand side. By (9.8),

$$\begin{aligned}\| s(T)p(T_m)z - s(T)e_0 \| \leqq & \; \| s(T) \| \left(\frac{1}{a_m} + \frac{1}{a_{n-1}}\right) \\ & \leqq |s| \cdot 2^{\deg s}\left(\frac{1}{a_m} + \frac{1}{a_{n-1}}\right) \qquad (\text{since } \| T \| \leqq 2 \text{ p.d.}) \\ & \leqq n \cdot 2^n \left(\frac{1}{a_m} + \frac{1}{a_{n-1}}\right) \\ & \leqq a_{n-1}^{-1/2} \qquad \text{for all } n \geqq 2 \quad \text{p.d.}\end{aligned} \tag{9.14}$$

Now similarly

$$\| p(T)s(T)z - s(T)p(T_m)z \| \leqq n \cdot 2^n \cdot \| (p(T) - p(T_m))z \| . \tag{9.15}$$

For each $r \in \mathbf{N}$,

$$\begin{aligned}T^r - T_m^r = T^{r-1}(T - T_m) + T^{r-2}(T - T_m)T_m \\ + T^{r-3}(T - T_m)T_m^2 + \cdots + (T - T_m)T^{r-1},\end{aligned}$$

hence

$$\| (T^r - T_m^r)z \| \leqq 2^r \cdot \max_{0 \leqq t \leqq \mu_m} \| (T - T_m)T_m^t z \|,$$

so by (9.6)

$$\| (p(T) - p(T_m))z \| \leqq 2^{\mu_m} \cdot \max_{0 \leqq t \leqq \mu_m} \| (T - T_m)T_m^t z \| \cdot | p |.$$

It is easily checked that for $s \leqq \mu_m$,

$$(T - T_m)T_m^s z = z_{\mu_m - s} e_{1+\mu_m},$$

a vector of norm at most

$$| z | \cdot 2^{-(b_m/2-1-\mu_m)/\sqrt{b_m}} \leqq a_m M_1(m, a_m) \cdot 2^{-(b_m/2-1-\mu_m)/\sqrt{b_m}}$$

by (2.8.4) and the definition of $K_{n,m}$ and $M_1(m, a_m)$; hence,

$$\| (p(T_m) - p(T))z \| \leqq 2^{\mu_m} \cdot a_m M_1(m, a_m) \cdot 2^{-(b_m/2-1-\mu_m)/\sqrt{b_m}} \cdot N_1(m, a_m),$$

so substituting in (9.15) we have

$$\begin{aligned} & \| s(T)p(T)z - s(T)p(T_m)z \| \\ (9.16) \qquad & \leqq n \cdot 2^{\mu_m + n} \cdot a_m \cdot M_1(m, a_m) \cdot N_1(m, a_m) \cdot 2^{-(b_m/2-1-\mu_m)/\sqrt{b_m}} \\ & < \frac{1}{b_m} \qquad \text{p.}d. \end{aligned}$$

Now

$$\| (q(T) - p(T)s(T))z \| = \left\| p(T)s(T)\left(\frac{T^{b_m}}{b_m} - I\right) z \right\|.$$

If $a_m \leqq i \leqq m a_m$ then

$$\left(\frac{T^{b_m}}{b_m} - I\right) e_i = \frac{1}{b_m} \cdot f_{i+b_m}$$

by (2.8.3); hence

$$(9.17) \qquad \left\| p(T)s(T)\left(\frac{T^{b_m}}{b_m} - I\right) e_i \right\| \leqq \frac{1}{b_m} \| p(T)s(T) \|.$$

If, on the other hand, $0 \leqq i \leqq a_m$ then, since $m \geqq 3$ (9.1), and since $t^{a_m} \mid p(T)$ (9.7), we have

$$p(T)s(T)\left(\frac{T^{b_m}}{b_m}-I\right)e_i = p_0(T)s(T)\left(\frac{T^{b_m}}{b_m}-I\right)e_{i+a_m}$$

(where $p_0(t)=t^{-a_m}p(T)$) and, as before, we find that $a_m \leqq i+a_m \leqq \mu_m$ so

$$\left\|\left(\frac{T^{b_m}}{b_m}-I\right)e_{i+a_m}\right\| = \frac{1}{b_m},$$

$$\left\| p(T)s(T)\left(\frac{T^{b_m}}{b_m}-I\right)e_i\right\| \leqq \frac{1}{b_m}\ \| p_0(T)s(T)\| . \tag{9.18}$$

Combining (9.17) and (9.18) we find that for all $z \in F_{\mu_m}$,

$$\left\| p(T)s(T)\left(\frac{T^{b_m}}{b_m}-I\right)z\right\| \leqq \frac{1}{b_m}\cdot |z|\cdot \max(\| p_0(T)s(T)\|,\ \| p(T)s(T)\|).$$

If z is our usual $Q_m^0(x)$ then

$$|z| \leqq M_1(m,a_m)\cdot \| z\| \leqq a_m\cdot M_1(m,a_m),$$

and hence $\| T\| \leqq 2$ p.$\boldsymbol{d}$., the right-hand side is at most

$$\frac{1}{b_m}\cdot a_m M_1(m,a_m)|p|\cdot|s|\cdot 2^{\deg p+\deg s} \leqq \frac{1}{b_m}\cdot a_m M_1(m,a_m)\cdot 2^{n+\mu_m}\cdot N_1(m,a_m)\cdot n$$

$$\leqq \frac{1}{\sqrt{b_m}} \quad \text{p.}\boldsymbol{d}.;$$

thus

$$\| (q(T)-p(T)s(T))z\| \leqq \frac{1}{\sqrt{b_m}}. \tag{9.19}$$

Consider now the term $q(T)(\pi_S-\pi_H)x$ on the right of (9.13). This vector is in $\mathrm{lin}\{f_i : \mu_m < i \leqq \nu_m\}$. We prove a further lemma concerning the action of T on such vectors.

LEMMA 9.20. P.$\boldsymbol{d}$., *the following is true. For all* $m \in \mathbf{N}$,

$$\| T^{a_m+b_m}|_{\mathrm{lin}\{f_i:\mu_m<i\leqq\nu_m\}}\| \leqq 2.$$

PROOF. By Definition 1.5, $\{f_i : \mu_m < i \leqq \nu_m\} \subset \{g_i\}$ so these f_i are the unit vector basis of a copy of l_1^N in $(F,\|\cdot\|)$ (where $N=\nu_m-\mu_m$). Hence, our statement is equivalent to

$$\| T^{a_m+b_m}f_i\| < 2 \qquad (\mu_m < i \leqq \nu_m). \tag{9.20.1}$$

This statement we check case by case.

Case 1. If for some r, $1 \leqq r \leqq m$, we have

$$i \in [r(a_m + b_m), ma_m + rb_m].$$

Then, by (2.8.3), $f_i = e_i - b_m e_{i-b_m}$.

Case 1a. If $r < m$ and $i \leqq (m-1)a_m + rb_m$, then (2.8.3) also gives

$$f_{i+a_m+b_m} = e_{i+a_m+b_m} - b_m e_{i+a_m}$$

hence

$$T^{a_m+b_m} f_i = f_{i+a_m+b_m}, \qquad \| T^{a_m+b_m} f_i \| = 1.$$

Case 1b. If $r < m$, $i > (m-1)a_m + rb_m$, writing $j = ma_m + (r+1)b_m$ we have

$$f_j = e_j - b_m e_{j-b_m} \qquad \text{(by (2.8.3))}$$

so since $j < i + a_m + b_m \leqq j + a_m$, for some $\alpha \in (0, a_m]$ we have

$$T^{a_m+b_m} f_i = T^\alpha f_j,$$

so

$$\| T^{a_m+b_m} f_i \| \leqq 2^{a_m-1} \| Tf_j \|.$$

As in Case 3b of the proof of Lemma 4, we note that Definition 2.8 gives

$$Tf_j = \varepsilon_1 f_{j+1} - b_m \varepsilon_2 \cdot f_{j+1-b_m}$$

for every j of the form $ma_m + rb_m$ $(1 \leqq r \leqq m)$, where

$$\varepsilon_2 = 2^{(ma_m+1-b_m/2)/\sqrt{b_m}}, \qquad \varepsilon_1 = \begin{cases} \varepsilon_2, & r < m, \\ 2^{(1+v_m-c_m/2)/\sqrt{c_m}}, & r = m. \end{cases}$$

Hence

$$\| Tf_j \| \leqq 1/b_m^2 \qquad \text{p.}\boldsymbol{d}.$$

and

$$\| T^{a_m+b_m} f_i \| \leqq 2^{a_m-1}/b_m^2 < 1/b_m \qquad \text{p.}\boldsymbol{d}.$$

Case 1c. If $r = m$, then $i = v_m$ and

$$T^{a_m+b_m} f_i = e_{v_m+a_m+b_m} - b_m e_{v_m+a_m}.$$

P.$\boldsymbol{d}$., $v_m + a_m + b_m < c_m$ so (2.8.6) gives

$$\| e_{v_m+a_m} \| = 2^{-(c_m/2-v_m-a_m)/\sqrt{c_m}} < 1/c_m \quad \text{p.}\boldsymbol{d}.,$$

$$\| e_{v_m+a_m+b_m} \| = 2^{-(c_m/2-v_m-a_m-b_m)/\sqrt{c_m}} < 1/c_m \qquad \text{p.}\boldsymbol{d}.$$

P.*d*., then,

$$\| T^{a_m+b_m} f_i \| \leqq \frac{1+b_m}{c_m} < 1.$$

Case 2. If for some r, $1 \leqq r \leqq m$, we have

$$i \in (ma_m + (r-1)b_m, r(a_m+b_m)).$$

Then by (2.8.4),

$$f_i = 2^{(h-i)/\sqrt{b_m}} e_i.$$

Case 2*a*. If $r < m$, then $i + a_m + b_m \in (ma_m + rb_m, (r+1)(a_m+b_m))$, so (2.8.4) conveniently gives

$$f_{i+a_m+b_m} = 2^{(h'-i-a_m-b_m)/\sqrt{b_m}} e_{i+a_m+b_m} \qquad (h' = h + b_m)$$
$$= 2^{(h-i-a_m)/\sqrt{b_m}} e_{i+q_m+b_m}.$$

Hence

$$\| T^{a_m+b_m} f_i \| = \| 2^{a_m/\sqrt{b_m}} f_{i+a_m+b_m} \| = 2^{a_m/\sqrt{b_m}} < 2 \qquad \text{p.}\boldsymbol{d}.$$

Case 2*b*. If $r = m$, then $i + a_m + b_m \in (v_m, v_m + a_m + b_m)$, hence since $\| T \| \leqq 2$, we have

$$\| T^{a_m+b_m} f_i \| = 2^{(h-i)/\sqrt{b_m}} \| e_{i+a_m+b_m} \|$$
$$\leqq 2^{((h-i)/\sqrt{b_m}+a_m+b_m)} \| e_{1+v_m} \|.$$

As will be well known by now, (2.8.6) gives

$$\| e_{1+v_m} \| = 2^{(1+v_m-c_m/2)/\sqrt{c_m}} < 1/c_m^2 \qquad \text{p.}\boldsymbol{d}.,$$

hence

$$\| T^{a_m+b_m} f_i \| < 2^{(h/\sqrt{b_m}+a_m+b_m)}.\frac{1}{c_m^2} < \frac{1}{c_m} \qquad \text{p.}\boldsymbol{d}.$$

Cases 1 and 2 cover all $i \in (\mu_m, v_m]$, so Lemma 9.20 is proved.

We apply our lemma in the following way:

$$\| q(T)(\pi_S - \pi_H)x \|$$
$$= \left\| \frac{T^{b_m}}{b_m} p(T)s(T)(\pi_S - \pi_H)x \right\|$$

$$= \left\| \frac{T^{a_m+b_m}}{b_m} \cdot p_0(T)s(T)(\pi_S - \pi_H)x \right\|$$

$$\leqq \frac{1}{b_m} \cdot \| s(T) \| \; \| p_0(T) \| \cdot 2 \cdot \| (\pi_S - \pi_H)x \|$$

by Lemma 9.20, since $(\pi_s - \pi_H)x \in \operatorname{lin}\{ f_i : \mu_m < i \leqq \nu_m \}$;

$$\leqq \frac{2}{b_m} \| s(T) \| \; \| p_0(T) \| \qquad \text{since } \| x \| = 1, \quad \| \pi_S - \pi_H \| = 1$$

$$\leqq (2^{1+\deg s + \deg p_0} | s | \cdot | p |)/b_m \qquad \text{since } \| T \| \leqq 2$$

$$\leqq (2nN_1(m, a_m) \cdot 2^{n+\mu_m})/b_m$$

by the various estimates from (9.3.1) to (9.6)

$$< \frac{1}{\sqrt{b_m}} \qquad \text{p.}\boldsymbol{d}. \tag{9.20.2}$$

since $n \leqq m$.

Furthermore, since $\| T \| \leqq 2$,

$$\| (\bar{p}_{k,\nu_m} - q)(T)(z + (\pi_S - \pi_H)x) \|$$

$$\leqq | \bar{p}_{k,\nu_m} - q | \cdot 2^{\deg(\bar{p}_{k,\nu_m} - q)} \| z + (\pi_S - \pi_H)x \|$$

$$\leqq 4^{-\nu_m} \cdot 2^{\nu_m} \cdot (a_m + 1)$$

by (9.2), (9.3), (3.2), (9.11), (2.1), (9.12), and because $\pi_S - \pi_H$ is a contraction;

$$\leqq \frac{1}{b_m} \qquad \text{p.}\boldsymbol{d}. \tag{9.21}$$

For each i, $0 \leqq i \leqq \nu_m$, (2.8.5) gives

$$(T^{c_m^k} - \bar{p}_{k,\nu_m}(T))e_i = c_m^{-1} f_{i+c_m^k}$$

hence

$$\| (T^{c_m^k} - \tilde{p}_{k,v_m}(T))(z + (\pi_S - \pi_H)x) \| \leqq c_m^{-1} \cdot | z + (\pi_S - \pi_H)x |$$
$$\leqq c_m^{-1} M_2(m, b_m) \cdot \| z + (\pi_S - \pi_H)x \|$$
$$\leqq c_m^{-1} M_2(m, b_m) \cdot (a_m + 1)$$

(since $z \in K_{n,m}$, $\pi_S - \pi_H$ is a contraction, and $z + (\pi_S - \pi_H)x \in F_{v_m}$)

$$\leqq \frac{1}{b_m} \qquad \text{p.}\boldsymbol{d}. \tag{9.22}$$

By Lemma 8,

$$\| T^{c_m^k} \circ \pi_H \| < 6,$$

where

$$H = \{ f_i : v_n < i \leqq \xi_n \} \subset \Delta_m.$$

Hence

$$T^{c_m^k} \circ \pi_H = T^{c_m^k} \circ \pi_H \circ \pi_{\Delta_m}$$

so

$$\| T^{c_m^k} \pi_H(x) \| \leqq 6 \| \pi_{\Delta_m}(x) \| . \tag{9.23}$$

Lastly we estimate $\| T^{c_m^k} \circ (I - R_m^0)x \|$ as follows. Write

$$x = y_1 + y_2 \qquad (y_1 \in l_1, y_2 \in W)$$

and in view of (5.0.2) we have

$$\| T^{c_m^k} \circ (I - R_m^0) y_2 \| \leqq \frac{1}{a_{m+1}} \| y_2 \| \leqq \frac{1}{a_{m+1}} .$$

Moreover, since

$$R_m^0 f_i = f_i \qquad (0 \leqq i \leqq \mu_m)$$

we have

$$T^{c_m^k} \circ (I - R_m^0) y_1 = T^{c_m^k} \circ (I - R_m^0) \pi_{\Delta_m} y_1$$

so by Lemma 5 with $\eta = 1$, p.$\boldsymbol{d}$. we have

$$\| T^{c_m^k} \circ (I - R_m^0) y_1 \| \leqq 2 \| \pi_{\Delta_m} y_1 \| = 2 \| \pi_{\Delta_m}(x) \| .$$

Thus,

$$\| T^{c_m^k} \circ (I - R_m^0)x \| \leqq \frac{1}{a_{m+1}} + 2 \| \pi_{\Delta_m}(x) \| . \tag{9.24}$$

Substituting (9.14), (916), (9.19), (9.20.2), (9.21), (9.22), (9.23) and (9.24) into (9.13), we have

$$\| T^{c_m^k}x - s(T)e_0 \| \leqq 8 \| \pi_{\Delta_m}(x) \| + a_{n-1}^{-1/2} + \frac{5}{b_m} + \frac{2}{\sqrt{b_m}} + \frac{1}{a_{m+1}}$$

$$\leqq 8 \| \pi_{\Delta_m}(x) \| + 2a_{n-1}^{-1/2} \qquad \text{p.}\boldsymbol{d}.,$$

as required. This completes the proof of Lemma 9.

§10. Proof of the main result

THEOREM 10. *$T:(X, \| \cdot \|) \to (X, \| \cdot \|)$ is hypercyclic.*

PROOF. Let $x, y \in X$ with $x \neq 0$, and let $\varepsilon > 0$. We will exhibit an M such that

$$\| T^M x - y \| < \varepsilon.$$

Without loss of generality $\| x \| = 1$. e_0 is obviously cyclic for T so choose a polynomial s such that

$$\| s(T)e_0 - y \| < \varepsilon/3. \tag{10.1}$$

Choose $N \in \mathbf{N}$ so large that $\deg s \leqq N$, $|s| \leqq N$, and $2a_N^{-1/2} < \frac{1}{3}\varepsilon$. Write $\eta = \varepsilon/24$. By Lemma 6 there are $m, n \in \mathbf{N}$ with $m > n > N$ such that

$$Q_m^0(x) \in K_{n,m}$$

and

$$\| \pi_{\Delta_m}(x) \| < \eta.$$

By Lemma 9 there is a k, $1 \leqq k \leqq M(\nu_m)$, such that

$$\| T^{c_m^k}x - s(T)e_0 \| \leqq 8 \| \pi_{\Delta_m}(x) \| + 2a_{n-1}^{-1/2}$$

$$\leqq 8\eta + 2a_N^{-1/2} < \tfrac{2}{3}\varepsilon.$$

By (10.1),

$$\| T^{c_m^k}x - y \| < \varepsilon.$$

So T is indeed hypercyclic.

§11. Notes

The ‘without loss of generality’ clause at the start of §1 is explained in detail in [7], indeed it is essentially just a quote from [3]. Operators without invariant subspaces were originally found by Enflo [2] and Read [4]; Enflo's proof was simplified by Beauzamy [1], and Read's proof was simplified [6] in such a fashion as to give an operator on l_1; the existence of such an operator was first shown in [5].

We are indebted to B. Beauzamy for drawing our attention to the question of whether hypercyclic operators existed. His example in [1] has the slightly weaker 'supercyclic' property, i.e., for each nonzero vector x, the set of scalar multiples of vectors $T^r(x)$ is dense in the underlying space.

The author would like to thank the mathematicians at the Hebrew University of Jerusalem for their hospitality during the fall semester '86, during which most of the work for this paper was accomplished.

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